MATH 235: Linear Algebra 2 for Honours Mathematics

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To the reader

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Chapter 1

Abstract Vector Spaces

1.1 The Definition of a Vector Space

Linear algebra is the study of vector spaces. Before we formally define a vector space, let's introduce some familiar examples of vector spaces. As you go through each example, pay close attention to the similarities between each.

Example 1.1.1 The vector space $\mathbb{R}^n$ is given by

$$\mathbb{R}^n = \left\{ \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} : a_i \in \mathbb{R} \text{ for all } i \right\}.$$

Addition and scalar multiplication are given by

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix} \quad \text{and} \quad \alpha \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \alpha a_1 \\ \vdots \\ \alpha a_n \end{bmatrix} \quad (\text{for } \alpha \in \mathbb{R}).$$

The intuitive picture that is helpful to have in mind are the cases of $\mathbb{R}^2$ and $\mathbb{R}^3$ that you are familiar with from previous courses. You can picture $\mathbb{R}^2$ as the Cartesian plane, and $\mathbb{R}^3$ as 3-dimensional space. In both of these vector spaces, you know how vector addition and scalar multiplication work, and intuitively, it's the same for $\mathbb{R}^n$. Although $\mathbb{R}^n$ is an n-dimensional vector space, it is usually helpful to use the visual imagery of $\mathbb{R}^2$ and $\mathbb{R}^3$.

In this course, we will study both real and complex vector spaces. The previous example is the most basic real vector space. There is an analogous complex vector space.

Example 1.1.2 The vector space $\mathbb{C}^n$ is given by

$$\mathbb{C}^n = \left\{ \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} : a_i \in \mathbb{C} \text{ for all } i \right\}.$$

Addition and scalar multiplication are given by

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix} \quad \text{and} \quad \alpha \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \alpha a_1 \\ \vdots \\ \alpha a_n \end{bmatrix} \quad (\text{for } \alpha \in \mathbb{C})$$

just like in the previous example. Notice however that we now allow ourselves to use *complex scalars* $\alpha \in \mathbb{C}$. The vector space $\mathbb{C}^n$ might appear to be even more challenging to visualize than $\mathbb{R}^n$. However, the algebra works just the same. In this course we will learn how our intuition from $\mathbb{R}^2$ and $\mathbb{R}^3$ will allow us to discover properties of $\mathbb{C}^n$. This is one of the strengths of linear algebra.

We will use the short-hand notation $\mathbb{F}$ (for "field") to denote either $\mathbb{R}$ or $\mathbb{C}$ when we do not wish to distinguish between them. This is convenient because many (though not at all!) of our results work equally well over both $\mathbb{R}$ and $\mathbb{C}$.

Example 1.1.3 The vector space $\mathbb{F}^n$ is given by

$$\mathbb{F}^n = \left\{ \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} : a_i \in \mathbb{F} \text{ for all } i \right\}.$$

Addition and scalar multiplication are given by

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix} \quad \text{and} \quad \alpha \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \alpha a_1 \\ \vdots \\ \alpha a_n \end{bmatrix} \quad (\text{for } \alpha \in \mathbb{F}).$$

If $\mathbb{F} = \mathbb{R}$, then $\mathbb{F}^n$ is the real vector space $\mathbb{R}^n$ from the first example; and if $\mathbb{F} = \mathbb{C}$, then $\mathbb{F}^n = \mathbb{C}^n$. Be mindful that the scalars α are always chosen to be in the appropriate $\mathbb{F}$. So when we are working with $\mathbb{F}^n = \mathbb{R}^n$, we only use real scalars $\alpha \in \mathbb{R}$.

Example 1.1.4 The vector space $\mathcal{P}_n(\mathbb{F})$ is the set of polynomials of degree at most n with coefficients in $\mathbb{F}$. That is

$$\mathcal{P}_n(\mathbb{F}) = \{a_0 + a_1 x + \cdots + a_n x^n : a_i \in \mathbb{F} \text{ for all } i\}$$

with addition and scalar multiplication defined by

$$(a_0 + a_1 x + \cdots + a_n x^n) + (b_0 + b_1 x + \cdots + b_n x^n) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n$$

and

$$\alpha(a_0 + a_1 x + \cdots + a_n x^n) = (\alpha a_0) + (\alpha a_1)x + \cdots + (\alpha a_n)x^n \quad (\alpha \in \mathbb{F})$$

respectively.

If a coefficient is 0, we usually omit it. So instead of writing $2 + 3x + 0x^2 + 4x^3 + 0x^5$, we'll simply write $2 + 3x + 4x^3$.

For example, $1+2x-3x^2 \in \mathcal{P}_2(\mathbb{R})$ and $1+(2+i)x-x^5 \in \mathcal{P}_5(\mathbb{C})$. (Both these polynomials are also in $\mathcal{P}_{10}(\mathbb{C})$, if we pretend they are missing some 0 coefficients.) We have

$$(4+7x)+(1+x^2) = 5+7x+x^2 \quad \text{and} \quad 25i(1+2ix^3) = 25i-50x^3.$$

You may be used to thinking of polynomials as functions. In the context of this course, don't! Although it is sometimes useful to evaluate a polynomial at a certain number, in this course, polynomials are not functions. They are simply objects which you can add together and multiply by scalars.

Example 1.1.5 The vector space of m by n matrices with entries in $\mathbb{F}$ is given by

$$M_{m\times n}(\mathbb{F}) = \left\{ \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} : a_{ij} \in \mathbb{R} \text{ for all } i,j \right\}.$$

Addition and scalar multiplication are given by matrix addition and scalar multiplication of matrices as usual. So, for example, in $M_{2\times 2}(\mathbb{R})$,

$$\begin{bmatrix} 2 & 5 \\ 7 & \pi \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 8 & \pi+1 \end{bmatrix} \quad \text{and} \quad \sqrt{2}\begin{bmatrix} 2 & 5 \\ 7 & \pi \end{bmatrix} = \begin{bmatrix} 2\sqrt{2} & 5\sqrt{2} \\ 7\sqrt{2} & \pi\sqrt{2} \end{bmatrix}.$$

Example 1.1.6 The vector space of real-valued continuous functions on the interval $[0,1]$ is denoted by

$$\mathcal{C}([0,1]) = \{f\colon [0,1] \to \mathbb{R} : f \text{ is continuous on } [0,1]\}.$$

Addition and scalar multiplication are defined by

$$(f+g)(x) = f(x)+g(x) \quad \text{and} \quad (\alpha f)(x) = \alpha(f(x)).$$

This example is a little trickier. Here, the 'vectors' are continuous functions from $[0,1]$ to $\mathbb{R}$. When we add two functions, we get another function, and when we multiply a function by a scalar, we get another function. For example,

$$(\sin(x)) + (\cos(x)) = \sin x + \cos x \quad \text{and} \quad 3(x^2+1) = 3x^2+3.$$

Example 1.1.7 Here's a slightly more interesting example. Let V be the set of all lines in $\mathbb{R}^2$ with slope 1. Each such line has equation $y = x+d$ for some $d \in \mathbb{R}$. Addition and scalar multiplication in V are defined by

$$(y = x+d_1) + (y = x+d_2) = (y = x+(d_1+d_2)) \quad \text{and} \quad \alpha(y = x+d) = (y = x+\alpha d).$$

Now that we have seen a few examples of vector spaces, you might have a perfectly reasonable question in mind: *What is a vector space?* Before we give the formal definition, let's take a look at the similarities between all of these examples. They all come with a set of 'vectors' (even though sometimes these vectors can look a little unusual, like a straight line in $\mathbb{R}^2$ of slope 1) and some set of scalars ($\mathbb{R}$ or $\mathbb{C}$). Furthermore, there is a way to 'add' two vectors to get another vector, and to 'multiply' a vector by a scalar to get another vector. There is some other structure lurking in the background which is perhaps a little harder to notice just from these examples. Indeed, in each vector space there is a special vector (call it $\vec{0}$) with the property that $\vec{0} + \vec{v} = \vec{v}$ for all vectors $\vec{v}$ in the vector space.

Formally, we define a vector space as follows:

Definition 1.1.8

Vector Space Over $\mathbb{F}$, Vector Addition, Scalar Multiplication, Vector Space Axioms, Zero Vector, Additive Inverse

A **vector space over** $\mathbb{F}$ is a set V together with an operation $+\colon V \times V \to V$ (**vector addition**) so that

$$\text{for all } \vec{x}, \vec{y} \in V,\ \vec{x} + \vec{y} \in V,$$

and an operation $\cdot\colon \mathbb{F} \times V \to V$ (**scalar multiplication**) so that

$$\text{for all } s \in \mathbb{F} \text{ and } \vec{x} \in V,\ s \cdot \vec{x} \in V.$$

These operations must satisfy the following properties.

1. For all $\vec{x}, \vec{y}, \vec{z} \in V$, $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$.
2. There exists a vector $\vec{0} \in V$ such that, for all $\vec{x} \in V$, $\vec{0} + \vec{x} = \vec{x} + \vec{0} = \vec{x}$.
3. For all $\vec{x} \in V$, there exists a vector $-\vec{x} \in V$ such that $\vec{x} + (-\vec{x}) = (-\vec{x}) + \vec{x} = \vec{0}$.
4. For all $\vec{x}, \vec{y} \in V$, $\vec{x} + \vec{y} = \vec{y} + \vec{x}$,
5. For all $\vec{x} \in V$ and $s, t \in \mathbb{F}$, $s \cdot (t \cdot \vec{x}) = (st) \cdot \vec{x}$,
6. For all $\vec{x} \in V$ and $s, t \in \mathbb{F}$, $(s + t) \cdot \vec{x} = s \cdot \vec{x} + t \cdot \vec{x}$,
7. For all $\vec{x}, \vec{y} \in V$ and $s \in \mathbb{F}$, $s \cdot (\vec{x} + \vec{y}) = s \cdot \vec{x} + s \cdot \vec{y}$, and
8. $1 \cdot \vec{x} = \vec{x}$.

The above properties are called the **vector space axioms**. The vector $\vec{0}$ in axiom 2 is called **the zero vector** of V. The vector $-\vec{x}$ in axiom 3 is called **the additive inverse of** $\vec{x}$.

REMARKS

- It is important to understand that the operations $+$ and $\cdot$ above must be supplied as part of the definition of a vector space. You have the freedom to define any two functions $+\colon V \times V \to V$ and $\cdot\colon \mathbb{F} \times V \to V$ on any set V. However, you will only get a vector space if your functions satisfy the vector space axioms.
- Be careful to not confuse the abstract scalar multiplication $\cdot$ and the usual scalar multiplication in $\mathbb{F}$. For instance, in axiom 5, on the left side we perform scalar multiplication by t and then by s; while on the right side we perform scalar multiplication by the scalar st. In general, there is no reason why these should produce the same

result. However, our intuition for a vector space suggests that they *should*, and this is why we formally require this property as an axiom.

- In this section, we will always use $\cdot$ for our abstract scalar multiplication. However, this gets cumbersome very quickly, and so in later sections we will omit it and simply write expressions like $a\vec{x}$ instead of $a \cdot \vec{x}$.
- We will show below that, in a vector space V, there can be exactly one vector that satisfies axiom 2. It is therefore acceptable to call this vector *the* zero vector of V. Likewise, for each $\vec{x} \in V$, there will be exactly one vector $-\vec{x}$ that satisfies axiom 3. See Proposition 1.1.12.
- The notation $-\vec{x}$ chosen for the additive inverse of $\vec{x}$ is very suggestive. It resembles the scalar multiplication $(-1) \cdot \vec{x}$ of $-1 \in \mathbb{F}$ and $\vec{x} \in V$. However, since the definition above is very abstract and general, it is conceivable that $-\vec{x}$ and $(-1) \cdot \vec{x}$ might be different vectors. Happily, that is not the case! It turns out that the vector space axioms actually imply that $-\vec{x} = (-1) \cdot \vec{x}$. See Proposition 1.1.13.
- The vector $\vec{x} + (-\vec{y})$ will usually be written as $\vec{x} - \vec{y}$.

Example 1.1.9 Let's check that $\mathbb{R}^2$ with the usual definitions of addition and scalar multiplication is a vector space over $\mathbb{R}$.

To check axiom 1, let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, and $\vec{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ be three arbitrary vectors in $\mathbb{R}^2$. Then

$$(\vec{x} + \vec{y}) + \vec{z} = \begin{bmatrix} x_1 + y_2 \\ x_2 + y_2 \end{bmatrix} + \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 + z_1 \\ x_2 + y_2 + z_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 + z_1 \\ y_2 + z_2 \end{bmatrix} = \vec{x} + (\vec{y} + \vec{z})$$

therefore axiom 1 holds.

The vector $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ satisfies the properties of axiom 2. If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, then $-\vec{x} = \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix}$ satisfies $\vec{x} + (-\vec{x}) = (-\vec{x}) + \vec{x} = \vec{0}$, so 3 holds. For 7, let $s \in \mathbb{R}$ be an arbitrary scalar. Then

$$\begin{aligned} s \cdot (\vec{x} + \vec{y}) &= s \cdot \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix} \\ &= \begin{bmatrix} s(x_1 + y_1) \\ s(x_2 + y_2) \end{bmatrix} \\ &= \begin{bmatrix} sx_1 + sy_1 \\ sx_2 + sy_2 \end{bmatrix} \\ &= \begin{bmatrix} sx_1 \\ sx_2 \end{bmatrix} + \begin{bmatrix} sy_1 \\ sy_2 \end{bmatrix} \\ &= s \cdot \vec{x} + s \cdot \vec{y}. \end{aligned}$$

We leave the rest of the axioms as an exercise for you to check.

EXERCISE

Prove that axioms 4, 5, 6, and 8 hold for $\mathbb{R}^2$, implying that $\mathbb{R}^2$ is indeed a vector space (over $\mathbb{R}$).

EXERCISE

Go through each of Examples 1.1.1–1.1.7 and convince yourself that each of them does in fact give a vector space (over the appropriate $\mathbb{F}$). In particular, what is the zero vector in Example 1.1.7?

Here is a not very interesting—but important!—example of a vector space.

Example 1.1.10 The **zero vector space** is the vector space $V = \{\vec{0}\}$ consisting of precisely one vector. The definitions of addition and scalar multiplication are obvious:

$$\vec{0} + \vec{0} = \vec{0} \quad \text{and} \quad \alpha \cdot \vec{0} = \vec{0} \quad (\alpha \in \mathbb{F}).$$

It's immediately obvious that these definitions satisfy axioms 1 through 8, with $\vec{0}$ (of course) being the zero vector.

Here is an example of something that is *not* a vector space.

Example 1.1.11 Let V be the set of polynomials with coefficients in $\mathbb{C}$ of degree at most 1, with addition of vectors and scalar multiplication (by scalars in $\mathbb{C}$) given by

$$(a_1 + a_2 x) + (b_1 + b_2 x) = (a_1 + a_2) + (b_1 + b_2)x \quad \text{and} \quad \alpha \cdot (a_1 + a_2 x) = \alpha a_2 + \alpha a_1 x$$

respectively. Then V is *not* a vector space since $1 \cdot (3 + ix) = i + 3x$ so axiom 8 fails.

Proposition 1.1.12 Let V be a vector space over $\mathbb{F}$. Then

(a) The zero vector in V is *unique*. That is, if $\vec{z} \in V$ satisfies the property that $\vec{x} + \vec{z} = \vec{x}$ for all $\vec{x} \in V$, then it must be the case that $\vec{z} = \vec{0}$.

(b) Let $\vec{x} \in V$. The additive inverse of $\vec{x}$ is *uniquely determined by* $\vec{x}$. That is, if $\vec{y}$ satisfies the property that $\vec{x} + \vec{y} = \vec{y} + \vec{x} = \vec{0}$, then $\vec{y} = -\vec{x}$.

Proof: (a) Suppose there were two different zero vectors, let's call them $\vec{0}_1$ and $\vec{0}_2$. Since $\vec{0}_1$ is a zero vector (i.e. it satisfies the properties of axiom 2), we have

$$\vec{0}_1 + \vec{0}_2 = \vec{0}_2.$$

Similarly, since $\vec{0}_2$ is a zero vector we have

$$\vec{0}_1 + \vec{0}_2 = \vec{0}_1.$$

The left sides of the preceding equations are the same, so the right sides must be too. Thus, $\vec{0}_1 = \vec{0}_2$. Therefore, the zero vector is unique.

(b) The equations
$$\vec{x} + (-\vec{x}) = \vec{0} \quad \text{and} \quad \vec{x} + \vec{y} = \vec{0}$$
imply that
$$\vec{x} + (-\vec{x}) = \vec{x} + \vec{y}.$$
Now add $\vec{y}$ to both sides:
$$\vec{y} + (\vec{x} + (-\vec{x})) = \vec{y} + (\vec{x} + \vec{y}).$$
By axiom 1, this is the same as
$$(\vec{y} + \vec{x}) + (-\vec{x}) = (\vec{y} + \vec{x}) + \vec{y}.$$
From the property given in (b), we know that $\vec{y} + \vec{x} = \vec{0}$. So the above equation is simply
$$\vec{0} + (-\vec{x}) = \vec{0} + \vec{y}.$$
Hence $-\vec{x} = \vec{y}$, by axiom 2.

□

Proposition 1.1.13 Let V be a vector space over $\mathbb{F}$. Then

(a) $0 \cdot \vec{x} = \vec{0}$ for all $\vec{x} \in V$,

(b) $(-1) \cdot \vec{x} = \vec{-x}$ for all $\vec{x} \in V$, and

(c) $t \cdot \vec{0} = \vec{0}$ for all $t \in \mathbb{F}$.

EXERCISE

Prove Proposition 1.1.13.

Notice that part (a) of the previous Proposition gives us a quick way of determining the zero vector of a given vector space: we simply scalar multiply any vector by the scalar zero. For instance, if V is the vector space from Example 1.1.7, and if, say, we consider the vector $y = x + 1$, then using the definition of scalar multiplication in V we find that
$$\vec{0} = 0 \cdot (y = x + 1) = (y = x).$$
Similarly, we can use part (c) to quickly determine the additive inverse of any given vector.

Of course, for this to be a useful strategy, we need to be sure that V is in fact a vector space. There are situations where we can be sure of this without having to check each of the 8 axioms.

1.2 Subspaces

We know that if we consider just the plane $\{\begin{bmatrix} x & y & 0 \end{bmatrix}^T : x, y \in \mathbb{R}^2\}$ consisting of only the xy-coordinates of points in $\mathbb{R}^3$, then we can think of this as "a copy" of $\mathbb{R}^2$ living inside $\mathbb{R}^3$. This is an example of a subspace of $\mathbb{R}^3$. To make this idea precise, we first formally define a subspace.

Definition 1.2.1 **Subspace**

Let V be a vector space over $\mathbb{F}$ and $U \subseteq V$ a subset. We call U a **subspace** of V if U, endowed with the addition and scalar multiplication from V, is itself a vector space over $\mathbb{F}$.

Example 1.2.2

Every vector space V has two obvious subspaces: V itself and the subspace $\{\vec{0}\}$ consisting of the zero vector of V.

Example 1.2.3

Consider the subset $U \subseteq \mathcal{P}_2(\mathbb{F})$ given by $U = \{p \in \mathcal{P}_2(\mathbb{F}) : p(2) = 0\}$. First to get a feel for U, note that $x^2 + x - 6 \in U$ but $x^2 \notin U$.

We claim that U is a subspace of $\mathcal{P}_2(\mathbb{F})$. Let's check some of the axioms to convince ourselves.

First we have to check that the addition and scalar multiplication from $\mathcal{P}_2(\mathbb{F})$ make sense as addition and scalar multiplication in U. That is, we have to make sure that if we take two vectors in U and add them together, we get a vector in U, and that every scalar multiple of a vector in U is in U.

Suppose $p, q \in U$ and $\alpha \in \mathbb{F}$. Then $(p+q)(2) = p(2) + q(2) = 0$ so $p + q \in U$. Furthermore, $(\alpha p)(2) = \alpha p(2) = 0$ so $\alpha p \in U$. Thus, addition and scalar multiplication make sense on U.

Since the addition and scalar multiplication on U is simply that from $\mathcal{P}_2(\mathbb{F})$, and $\mathcal{P}_2(\mathbb{F})$ is a vector space, axioms 1, 4, 5, 6, 7, and 8 obviously hold for U. Since that the zero vector $\vec{0} = 0x^2 + 0x + 0$ of $\mathcal{P}_2(\mathbb{F})$ is in U, we deduce that axiom 2 is satisfied. Finally, by part (b) of Proposition 1.1.13, $-p = (-1)p \in U$, so axiom 3 is satisfied. We may finally conclude that U is a vector space.

Checking that addition and scalar multiplication make sense on U *and* checking all 8 axioms is a little cumbersome. However, if you carefully examine the previous example, a lot of things came for free from the fact that $\mathcal{P}_2(\mathbb{F})$ was already a vector space. The next theorem allows us never to have to do that much work again, and simply check three things to check whether or not a subset of a vector space is a subspace or not.

Theorem 1.2.4

(The Subspace Test)

Let V be vector space over $\mathbb{F}$ and let U be a subset of V. Then U is a subspace of V if and only if the following three conditions hold.

(a) U is non-empty.

(b) For all $\vec{u}_1, \vec{u}_2 \in U$, $\vec{u}_1 + \vec{u}_2 \in U$. (We say that U is **closed under addition.**)

(c) For all $\alpha \in \mathbb{F}$ and for all $\vec{u} \in U$, $\alpha\vec{u} \in U$. (We say that U is **closed under scalar multiplication**.)

Proof: If U is a subspace, then (b) and (c) hold as part of being a definition of a subspace, and since all vector spaces have a zero vector, U must be non-empty.

Conversely, suppose (a), (b) and (c) hold for a subset U of V. Properties (b) and (c) imply that the addition and scalar multiplication from V restrict to addition and scalar multiplication on U. Axioms 1,4,5,6,7, and 8 hold since V is a vector space. For axiom 2, since U is non-empty, choose a vector $\vec{u} \in U$ and then note by Proposition 1.1.13, $0\vec{u} = \vec{0}$. Property (c) then implies that $\vec{0}$ is in U. Similarly, for axiom 3 let $\vec{u} \in U$. Then by Proposition 1.1.13 and property (c), $-\vec{u} = (-1)\vec{u} \in U$, completing the proof. □

Example 1.2.5 Prove that $U = \{p \in \mathcal{P}_2(\mathbb{F}) : p(2) = 0\}$ is a subspace of $\mathcal{P}_2(\mathbb{F})$.

Proof: By the Subspace Test, we only need to check three things.

1. Since $\vec{0} = 0x^2 + 0x + 0 \in U$, U is non-empty.
2. Let $p, q \in U$. Then $(p+q)(2) = p(2) + q(2) = 0$, so $p + q \in U$.
3. Let $p \in U$ and $\alpha \in \mathbb{R}$. Then $(\alpha p)(2) = \alpha p(2) = 0$ so $\alpha p \in \mathbb{R}$.

Therefore by the Subspace Test, U is a subspace of $\mathcal{P}_2(\mathbb{F})$. □

It is natural to ask now what kind of things aren't subspaces. If you study the proof of The Subspace Test, you will see that a subspace of a vector space V must contain the zero vector of V.

Corollary 1.2.6 Let V be a vector space over $\mathbb{F}$ and suppose that U is a subspace of V. Then $\vec{0} \in U$.

EXERCISE

Prove Corollary 1.2.6.

Example 1.2.7 Let $S = \{p(x) \in \mathcal{P}_2(\mathbb{F})\colon p(2) = 1\}$. Then S is not a subspace of $\mathcal{P}_2(\mathbb{F})$ because it does not contain $\vec{0} = 0 + 0x + 0x^2$, the zero vector of $\mathcal{P}_2(\mathbb{F})$.

Of course a subset of a vector space may contain $\vec{0}$ yet fail to be a subspace.

Example 1.2.8 Consider the subset

$$L = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{F}) : a, b, c, d \in \mathbb{Z} \right\}.$$

of $M_{2\times 2}(\mathbb{F})$. This is not a subspace of $M_{2\times 2}(\mathbb{F})$ since $\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \notin L$ whereas $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \in L$. That is, L is not closed under scalar multiplication.

Now that we have studies some examples, an interesting question to think about is how subspaces can be created. One way is to take a set of vectors in your vector space, and then throw in everything else that needs to be there to make that subset a subspace! This is the same process that you have seen in $\mathbb{F}^n$. The following definition uses familiar terminology.

Definition 1.2.9

Span, Linear Combination

Let $S = \{\vec{v}_1, \ldots, \vec{v}_k\}$ be a subset of a vector space V. Define the **span** of S by

$$\operatorname{Span}(S) = \{t_1\vec{v}_1 + \cdots + t_k\vec{v}_k : t_1, \ldots, t_k \in \mathbb{F}\}.$$

A vector of the form $t_1\vec{v}_1 + \cdots + t_k\vec{v}_k$ is called a **linear combination** of the vectors $\vec{v}_1, \ldots, \vec{v}_k$.

By convention, we define the span of the empty set to be the set consisting of the zero vector: $\operatorname{Span}\varnothing = \{\vec{0}\}$.

REMARK

Notice that the previous definition only applies to finite subsets of V. If S is an infinite subset of V, then the span of S is defined to be the union of the spans of all finite subsets of S. Equivalently, the span of S is the set of all linear combinations of all finite collections of vectors in S.

In this course, we will not be making use of this more general definition. However, in more advanced treatments of linear algebra, this generalization plays an important role.

Example 1.2.10

In $\mathcal{P}_1(\mathbb{F})$, let $S = \{1+x, 1-x\}$. Then

$$\operatorname{Span}(S) = \{a(1+x) + b(1-x) : a, b \in \mathbb{F}\}.$$

There are of course many different descriptions of $\operatorname{Span}(S)$. For instance, we claim that

$$\operatorname{Span}(S) = \{c + dx : c, d \in \mathbb{F}\}.$$

Indeed, notice that

$$c + dx = \frac{c+d}{2}(1-x) + \frac{c-d}{2}(1+x).$$

(Where did this seemingly magical expression come from? You may want to review how to solve systems of linear equations!) This shows that $c + dx \in \operatorname{Span}(S)$, and therefore that $\{c + dx : c, d \in \mathbb{F}\} \subseteq \operatorname{Span}(S)$. Conversely, to show the reverse containment $\supseteq$, observe that

$$a(1+x) + b(1-x) = (a+b)1 + (a-b)x.$$

This shows that an arbitrary element of $\operatorname{Span}(S)$ belongs to $\{c + dx : c, d \in \mathbb{F}\}$.

EXERCISE

Consider the system

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}.$$

What does this have to do with the previous example?

Let's prove now that taking the span of some vectors does actually result in a subspace. The proof is exactly as it was for $\mathbb{F}^n$. This is a common theme of the subject. Any result that we can prove for $\mathbb{F}^n$ using only the vector space structure of $\mathbb{F}^n$ can usually be carried over word for word to the setting of an abstract vector space over $\mathbb{F}$. Indeed, this is one motivation for even defining abstract vector spaces! We can now prove one result that simultaneously applies to a wide variety of different-looking spaces, such as $\mathbb{F}^n$, $M_{m\times n}(\mathbb{F})$, $\mathcal{P}_n(\mathbb{F})$, etc.

Proposition 1.2.11

Let $S = \{\vec{v}_1, \dots, \vec{v}_k\}$ be a subset of a vector space V. Then $\operatorname{Span}(S)$ is a subspace of V.

Proof: Since $\vec{0} = 0\vec{v}_1 + \cdots + 0\vec{v}_k$, $\vec{0} \in \operatorname{Span}(S)$ so $\operatorname{Span}(S)$ is non-empty. Suppose $\vec{x}, \vec{y} \in \operatorname{Span}(S)$, and let $\vec{x} = t_1\vec{v}_1 + \cdots + t_k\vec{v}_k$ and $y = s_1\vec{v}_1 + \cdots + s_k\vec{v}_k$ for elements $t_1, \dots, t_k, s_1, \dots, s_k \in \mathbb{F}$. Then

$$\vec{x} + \vec{y} = (t_1 + s_1)\vec{v}_1 + \cdots + (t_k + s_k)\vec{v}_k$$

so $\vec{x} + \vec{y} \in \operatorname{Span}(S)$. Finally, let $\vec{x} \in \operatorname{Span}(S)$ be as above, and let $\alpha \in \mathbb{F}$. Then $\alpha\vec{x} = (\alpha t_1)\vec{v}_1 + \cdots + (\alpha t_k)\vec{v}_k$ and since $\alpha t_i \in \mathbb{F}$ for all i, $\alpha\vec{x} \in \operatorname{Span}(S)$. Therefore, by the Subspace Test, $\operatorname{Span}(S)$ is a subspace of V. $\square$

1.3 Bases and Dimension

We now shift our focus to formalising the notion of dimension. Intuitively we know that $\mathbb{R}^2$ is a 2-dimensional space, because there are 2 different directions one can travel in, and no more. We may also have an idea that $\mathbb{R}^2$ is 2-dimensional since every vector is determined by 2 pieces of information (the x and y coordinate). Similarly, we may guess that $\mathbb{R}^n$ would be an n-dimensional vector space, and we would be correct! However, this geometric intuition fails us when thinking about other vector spaces. For example, what is the dimension of $\mathbb{C}^2$, or $\mathcal{P}_3(\mathbb{R})$, or $\mathcal{C}([0,1])$?

As you've learned in a previous course, the key to defining a useful notion of dimension is to first define "basis." The definition of basis for $\mathbb{F}^n$ carries over without change to the setting of an abstract vector space. Recall that a basis for $\mathbb{F}^n$ is a linearly independent spanning set for $\mathbb{F}^n$. Thus, we must begin by defining these concepts in this new abstract setting.

1.3.1 Linear Independence, Spanning Sets, and Bases

Definition 1.3.1
Spanning Set, Spans

A set of vectors $S = \{\vec{v}_1, \dots, \vec{v}_k\}$ in a vector space V is a **spanning set** for V if $\operatorname{Span}(S) = V$. We also say that S **spans** V.

Intuitively, a set of vectors span a vector space if every vector in that vector space can be obtained from those vectors. More precisely, every vector in the vector space is a linear combination of those from the spanning set.

Example 1.3.2

The set $B = \{1, x\}$ is a spanning set for $\mathcal{P}_1(\mathbb{F})$. Indeed, by definition

$$\mathcal{P}_1(\mathbb{F}) = \{a + bx \colon a, b \in \mathbb{F}\} = \operatorname{Span}(B).$$

In Example 1.2.10, we effectively showed that $S = \{1 - x, 1 + x\}$ is also a spanning set for $\mathcal{P}_1(\mathbb{F})$.

It is easy to see that $T = \{3 + 2x\}$ is not a spanning set for $\mathcal{P}_1(\mathbb{F})$. For instance, there is no way of writing the polynomial x as a multiple of $3 + 2x$.

A spanning set can sometimes have redundant information. For example, the sets

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

are both spanning sets for $\mathbb{R}^2$, but the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in the first set is redundant. Somehow this is because in the second set, the two vectors point in different directions, but in the first, the three do not. To formalize this, we introduce the notion of linear independence.

Definition 1.3.3

Linearly Independent, Linearly Dependent

A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ in a vector space V is **linearly independent** if the only solution to the equation

$$t_1 \vec{v}_1 + \cdots + t_k \vec{v}_k = \vec{0}$$

is $t_1 = \cdots = t_k = 0$. The set is **linearly dependent** otherwise.

By convention, the empty set $\varnothing$ is linearly independent.

REMARK

We can extend the definition of linear independence to infinite subsets of V by defining such a set to be linearly independent if all of its finite subsets are linearly independent. Just as for spanning sets, this more general definition will not play a role in our course.

Although this is the formal definition we are to work with, the intuition is that a linearly independent set is a set of vectors that all point in different directions.

Example 1.3.4

The set $\{1 + x, 1\}$ is linearly independent in $\mathcal{P}_1(\mathbb{C})$. To see this, set

$$0 = t_1(1 + x) + t_2(1) = (t_1 + t_2) + t_1 x.$$

Then equating the x coefficient gives us $t_1 = 0$, which then implies $t_2 = 0$. Therefore the only solution is $t_1 = t_2 = 0$, so the set is linearly independent.

Example 1.3.5

Since in $\mathbb{R}^2$,

$$-1\begin{bmatrix}1\\0\end{bmatrix}+\frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix},$$

the set $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix}\right\}$ is linearly dependent.

Sometimes it's not so easy to stare at a set of vectors and decide whether or not they are linearly independent. However, we do have some tools for solving simultaneous equations from a previous course to help us along the way!

Example 1.3.6

Is $\{x+x^2-2x^3, 2x-x^2+x^3, x+5x^2+3x^3\}$ linearly independent in $\mathcal{P}_3(\mathbb{R})$?

Solution:

To check, we want to solve the equation

$$a(x+x^2-2x^3)+b(2x-x^2+x^3)+c(x+5x^2+3x^3)=0$$

for a, b, c. Equating coefficients gives us the system of simultaneous equations

$$\begin{aligned} a+2b+c&=0\\ a-b+5c&=0\\ -2a+b+3c&=0. \end{aligned}$$

To solve such a system of equations, we plug the coefficients into an augmented matrix and row reduce! We get

$$\left[\begin{array}{ccc|c}1&2&1&0\\1&-1&5&0\\-2&1&3&0\end{array}\right]\longrightarrow\left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right].$$

Therefore the system of equations has exactly one solution, and that solution is $a=b=c=0$. Therefore the set is linearly independent.

If we have a spanning set that is linearly independent, then in some sense our spanning set is not redundant. Such sets are very special and deserve a name.

Definition 1.3.7
Basis

A **basis** for a vector space V is a linearly independent subset that spans V.

Theorem 1.3.8

Every vector space has a basis.

We will not prove Theorem 1.3.8. For the vector spaces that we will be studying in this course, it will be easy to write down explicit bases (see Section 1.3.3). Here are some examples.

- $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is the **standard basis** for $\mathbb{F}^n$.
- $\{1, x, x^2, \cdots, x^n\}$ is the **standard basis** for $\mathcal{P}_n(\mathbb{F})$.
- $\{E_{11}, E_{12}, \dots, E_{ij}, \dots, E_{nm}\}$, where E_{ij} is the $m \times n$ matrix with an entry of 1 in the (i, j)th position and 0s elsewhere, is the **standard basis** for $M_{m\times n}(\mathbb{F})$.
- The empty set $\varnothing$ is a basis for the zero vector space $\{\vec{0}\}$.
- $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$.
- $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ is a basis for $\mathbb{C}^3$.
- $\left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3i \end{bmatrix} \right\}$ is a basis for $\mathbb{C}^2$.
- $\{1 - x, 1 + x\}$ is a basis for $\mathcal{P}_1(\mathbb{R})$.
- $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for $M_{2\times 2}(\mathbb{R})$.

EXERCISE

Verify that the sets above are bases for the indicated vector spaces.

1.3.2 Dimension

Just as for $\mathbb{F}^n$, we will define the dimension of a vector space V to be the number of vectors in a basis for V. For this to make sense, we must first prove that all bases have the same size. As you examine our proof below, you should compare it to the proof of the same fact for $\mathbb{F}^n$ that you may have seen in a previous course.

Lemma 1.3.9 Let V be a vector space over $\mathbb{F}$ and suppose that $V = \text{Span}(\{\vec{v}_1, \dots, \vec{v}_n\})$. If $\{\vec{u}_1, \dots, \vec{u}_k\}$ is a linearly independent set in V, then $k \leq n$.

Proof: Since $\text{Span}(\{\vec{v}_1, \dots, \vec{v}_n\}) = V$, we have

$$\begin{aligned} \vec{u}_1 &= a_{11}\vec{v}_1 + \cdots + a_{1n}\vec{v}_n \\ &\vdots \\ \vec{u}_k &= a_{k1}\vec{v}_1 + \cdots + a_{kn}\vec{v}_n \end{aligned}$$

where $a_{ij} \in \mathbb{F}$ for all i and j. We will now aim to show that if $k > n$, then there is a solution to $t_1\vec{u}_1 + \cdots + t_k\vec{u}_k = \vec{0}$ where not all the t_i are 0. We have

$$\begin{aligned} &t_1\vec{u}_1 + \cdots + t_k\vec{u}_k \\ &\quad = t_1(a_{11}\vec{v}_1 + \cdots + a_{1n}\vec{v}_n) + \cdots + t_k(a_{k1}\vec{v}_1 + \cdots + a_{kn}\vec{v}_k) \\ &\quad = (a_{11}t_1 + a_{21}t_2 + \cdots + a_{k1}t_k)\vec{v}_1 + \cdots + (a_{1n}t_1 + \cdots + a_{kn}t_k)\vec{v}_n. \end{aligned}$$

Now, if $k > n$ the system of linear equations

$$\begin{aligned} a_{11}t_1 + \cdots + a_{k1}t_k &= 0 \\ &\vdots \\ a_{1n}t_1 + \cdots + a_{kn}t_k &= 0 \end{aligned}$$

has a solution where not all the t_i are 0. Consider such a solution. We then have

$$\begin{aligned} \vec{0} &= 0\vec{v}_1 + \cdots + 0\vec{v}_n \\ &= (a_{11}t_1 + \cdots + a_{k1}t_k)\vec{v}_1 + \cdots (a_{1n}t_1 + \cdots + a_{kn}t_k)\vec{v}_n \\ &= t_1\vec{u}_1 + \cdots + t_k\vec{u}_k \end{aligned}$$

contradicting the assumption that $\{\vec{u}_1, \ldots, \vec{u}_k\}$ is linearly independent. So $k \leq n$. □

Theorem 1.3.10 Suppose $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ and $\mathcal{C} = \{\vec{u}_1, \ldots, \vec{u}_k\}$ are both bases of a vector space V. Then $k = n$.

Proof: Since $\mathcal{B}$ spans V and $\mathcal{C}$ is linearly independent, $k \leq n$. However, since $\mathcal{C}$ spans V and $\mathcal{B}$ is linearly independent, $n \leq k$. Thus, $k = n$. □

Definition 1.3.11
Dimension

The **dimension** of a vector space V, denoted by $\dim(V)$, is the size of any basis for V.

Note that Theorem 1.3.10 shows that this definition makes sense.

REMARK

If there is no finite basis for a vector space V, then we say V is infinite-dimensional.

With the definition of dimension at our disposal, we can now talk about dimension with conviction! Here are four important examples:

- $\dim(\{\vec{0}\}) = 0$ since by convention $\varnothing$ is a basis for $\{\vec{0}\}$.
- $\dim(\mathbb{F}^n) = n$ since the standard basis has size n.
- $\dim(\mathcal{P}_n(\mathbb{F})) = n + 1$ since the standard basis has size $n + 1$.
- $\dim(M_{m\times n}(\mathbb{F})) = mn$ since the standard basis has size nm.

Example 1.3.12 Let $U = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{F}) : a+b+c+d=0 \right\}$. It is an exercise for you to check that U is a subspace of $M_{2\times 2}(\mathbb{F})$. We will now compute the dimension of U by finding a basis for U.

Note that every matrix in U is of the form $\begin{bmatrix} a & b \\ c & -a-b-c \end{bmatrix}$, so we can write every matrix in U as

$$a\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$$

so $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \right\}$ is a spanning set for U.

(Notice that this actually proves that U is a subspace of $M_{2\times 2}(\mathbb{F})$! Indeed, we have just shown that U is equal to a span of some vectors in $M_{2\times 2}(\mathbb{F})$. Now apply Proposition 1.2.11.)

We now check to see whether the above spanning set is linearly independent. Consider

$$t_1\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + t_2\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + t_3\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then by looking at the top left entry we have $t_1 = 0$; the top right gives $t_2 = 0$; and the bottom left gives $t_3 = 0$. Therefore $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \right\}$ is linearly independent and hence is a basis for U. Thus, $\dim(U) = 3$.

The next theorem is extremely useful in thinking about dimension. It formally proves things you already know in your heart to be true. Things like "You cannot have 4 linearly independent vectors in $\mathbb{R}^3$, there's just not enough space!" and "You can't span $M_{2\times 2}(\mathbb{C})$ with only 3 vectors, that's not enough because $\dim(M_{2\times 2}(\mathbb{C})) = 4$!" A sketch of the proof is provided, and you should fill in the details as an exercise. (If you get stuck, you can look at the proof in the case of $V = \mathbb{F}^n$ for inspiration.)

Theorem 1.3.13 Let V be an n-dimensional vector space over $\mathbb{F}$. Then

(a) A set of more than n vectors in V must be linearly dependent.

(b) A set of fewer than n vectors in V cannot span V.

(c) A set with exactly n vectors in V is a spanning set for V if and only if it is linearly independent.

Proof: Parts (a) and (b) are restatements of Lemma 1.3.9. Statement (c) follows from the two paragraphs in Section 1.3.3. □

Finally, we state an unsurprising result relating the dimension of a subspace to the dimension of the vector space that contains it.

Theorem 1.3.14 Let V be a finite-dimensional vector space over $\mathbb{F}$ and let W be a subspace of V. Then $\dim(W) \leq \dim(V)$ with equality if and only if $W = V$.

Proof: Since any basis for W can be extended to a basis for V, as described in Section 1.3.3 below, the inequality $\dim(W) \leq \dim(V)$ follows. Suppose now that $\dim(W) = \dim(V)$. Then according to Theorem 1.3.13(c), a basis $\mathcal{B}$ for W will automatically be a basis for V, since it is a linearly independent set of size $\dim(V)$. It follows that $V = \text{Span}(\mathcal{B}) = W$. Conversely, if $W = V$ then of course $\dim(W) = \dim(V)$. □

1.3.3 Obtaining Bases

There are many ways you could find a basis for a finite-dimensional vector space. Here are a couple of important ways.

1. **Extending a linearly independent subset**. Suppose you have a linearly independent subset $\{\vec{v}_1, \ldots, \vec{v}_k\}$ in a finite dimensional vector space V. If it is a spanning set, then you have a basis. If not, choose a vector $\vec{v}_{k+1}$ not in the span of $\{\vec{v}_1, \ldots, \vec{v}_k\}$. Then $\{\vec{v}_1, \ldots, \vec{v}_{k+1}\}$ must be linearly independent. If this new set spans, then it's a basis. If not, then repeat. This process must eventually stop since our vector space is finite-dimensional, and you will be left with a basis containing $\{\vec{v}_1, \ldots, \vec{v}_k\}$.

2. **Reducing an arbitrary finite spanning set**. Suppose you have a finite spanning set $\{\vec{v}_1, \ldots, \vec{v}_k\}$ for your vector space, and let's assume that it doesn't contain $\vec{0}$. If it is linearly independent, it is a basis! If not, you can write one of them, say v_i, as a linear combination of the others. Now $\text{Span}(\{\vec{v}_1, \ldots, \vec{v}_k\}) = \text{Span}(\{\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_k\})$, so $\{\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_k\}$ spans our vector space. If this new set is linearly independent, then it is a basis! If not, repeat to remove another vector. This process must eventually stop since we started with finitely many vectors in our spanning set. The final product will be a basis made up entirely out of vectors from our original spanning set.

In practice, most of our vector spaces will be given to us as subspaces of a familiar vector space V, like in Examples 1.2.5 and 1.3.12, where we were given a subspace U of V defined by some conditions. In such situations we can attempt to express the defining conditions for U in terms of a known basis for V (such as the "standard basis" in case V is one of $\mathbb{F}^n$, $\mathcal{P}_n(\mathbb{F})$ or $M_{m\times n}(\mathbb{F})$). This will allow us to determine a spanning set for U, after which we can apply method 2 above (if this spanning set isn't already a basis).

Example 1.3.15 To give an illustration of how this works, let's try to find a basis for the subspace

$$U = \{p \in \mathcal{P}_2(\mathbb{F}) : p(2) = 0\}$$

of $\mathcal{P}_2(\mathbb{F})$ from Example 1.2.5. First we must express the condition $p(2) = 0$ using the standard basis $\{1, x, x^2\}$ of $\mathcal{P}_2(\mathbb{F})$. We can write p as $p(x) = a + bx + cx^2$, in which case the condition $p(2) = 0$ becomes

$$a + 2b + 4c = 0.$$

This is a linear equation in three variables, and we can employ our usual method for solving it in terms of basic variables and free parameters. We can do this quickly: simply notice that $a = -2b - 4c$! Consequently,

$$\begin{aligned} p(x) &= a + bx + cx^2 \\ &= (-2b - 4c) + bx + cx^2 \\ &= b(-2 + x) + c(-4 + x^2). \end{aligned}$$

This shows that $\mathcal{B} = \{-2 + x, -4 + x^2\}$ is a spanning set for U. We'll leave it as an exercise for you to check that $\mathcal{B}$ is linearly independent, and is therefore a basis for U.

EXERCISE

Re-do Example 1.3.12 by following the process used in the previous example.

1.3.4 Coordinates with Respect to a Basis

Recall that in $\mathbb{R}^3$ you may have seen that the vector $\begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$ can be written as $3\vec{e}_1 + 2\vec{e}_2 + 4\vec{e}_3$, where $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is the standard basis for $\mathbb{R}^3$. You have seen this to mean that the vector $\begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$ can be found 3-units in the x-direction, 2 in the y, and 4 in the z.

In fact, once we have a basis for a vector space, we can think of this as a choice of axes, and we can write every vector as a coordinate vector in much the same way as we think about vectors in $\mathbb{R}^3$.

Example 1.3.16 Consider the vector $\vec{v} = 3 + 5x - 2x^2$ in $\mathcal{P}_2(\mathbb{R})$, and the bases $\mathcal{B} = \{1, x, x^2\}$ and $\mathcal{C} = \{1, 1 + x, 1 + x + x^2\}$ (as an exercise, prove $\mathcal{C}$ is a basis). Then $\vec{v} = 3(1) + 5(x) + (-2)(x^2)$ so we think of $\vec{v}$ as living at the coordinate $(3, 5, -2)$ with respect to the axes defined by $\mathcal{B}$. We also have $\vec{v} = -2(1) + 7(1 + x) + (-2)(1 + x + x^2)$ so, with respect to the axes determined by $\mathcal{C}$, we can think of $\vec{v}$ as living at the point $(-2, 7, -2)$. More formally, we can write the coordinate vectors of $\vec{v}$ with respect to $\mathcal{B}$ and $\mathcal{C}$ as

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} \quad \text{and} \quad [\vec{v}]_{\mathcal{C}} = \begin{bmatrix} -2 \\ 7 \\ -2 \end{bmatrix}$$

respectively. This gives us two different ways of looking at the same vector.

A natural question to ask is: does it even make sense to talk about coordinate vectors like this? Is it possible that the same vector has two different coordinate vectors with respect to the same basis? The answer is "no."

Lemma 1.3.17

Let V be a vector space, let $S = \{\vec{v}_1, \dots, \vec{v}_n\}$ be a subset of V, and let $U = \text{Span}(S)$. Then every vector in U can be expressed in a unique way as a linear combination of the vectors in S if and only if S is linearly independent.

Proof: Suppose every vector in U is expressed uniquely as a linear combination of the vectors in S. Then there is only one way to write

$$\vec{0} = t_1\vec{v}_1 + \cdots + t_k\vec{v}_k,$$

which is $t_1 = \cdots = t_k = 0$, so S is linearly independent. Conversely, suppose S is linearly independent and

$$t_1\vec{v}_1 + \cdots + t_k\vec{v}_k = s_1\vec{v}_1 + \cdots + s_k\vec{v}_k.$$

Rearranging we have $(t_1 - s_1)\vec{v}_1 + \cdots + (t_k - s_k)\vec{v}_k = \vec{0}$. Since S is linearly independent, this can only be true if $t_i = s_i$ for all i, completing the proof. □

If we apply this lemma to a basis of a vector space, we immediately get the following useful theorem.

Theorem 1.3.18

(Unique Representation Theorem)

Let V be a vector space and let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ be a basis of V. Then for all $\vec{v} \in V$ there exist unique scalar $x_1, \dots, x_n \in \mathbb{F}$ such that

$$\vec{v} = x_1\vec{v}_1 + \cdots + x_n\vec{v}_n.$$

We can thus unambiguously define the set of coordinates of a vector with respect to a given basis. But if we want to use these coordinates to form a *coordinate vector*, there is a small subtlety that must be addressed. The next example illustrates the issue.

Example 1.3.19

Let $\vec{v} = 2 - i + 4x - ix^2 \in \mathcal{P}_2(\mathbb{C})$. If $\mathcal{B} = \{1, x, x^2\}$ and $\mathcal{C} = \{1, x^2, x\}$ are bases for $\mathcal{P}_2(\mathbb{C})$, then the coordinates of $\vec{v}$ with respect to these bases are $\begin{bmatrix} 2-i \\ 4 \\ -i \end{bmatrix}$ and $\begin{bmatrix} 2-i \\ -i \\ 4 \end{bmatrix}$, respectively.

That is, the order of the basis vectors matters!

Definition 1.3.20
Ordered Basis

Let V be a vector space over $\mathbb{F}$. An **ordered basis for** V is a basis $\mathcal{B} = \{\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}\}$ for V together with a fixed ordering.

REMARK

When we refer to the set $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ as being *ordered*, we are indicating that $\vec{v_1}$ is the first element in the ordering, that $\vec{v_2}$ is the second, and so on.

Thus even though $\{\vec{v_1}, \vec{v_2}\}$ and $\{\vec{v_2}, \vec{v_1}\}$ are the same *set*, they are different from the point of view of orderings.

A basis $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ gives rise to $n!$ ordered bases, one for each possible ordering (permutation) of the vectors in the basis.

Definition 1.3.21
Coordinate Vector

Let $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be an *ordered* basis for a vector space V. If $\vec{x} \in V$ is written as

$$\vec{x} = x_1\vec{v}_1 + \cdots + x_n\vec{v}_n$$

then the **coordinate vector of $\vec{x}$ with respect to $\mathcal{B}$** is

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Example 1.3.22

Consider the ordered basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix} \right\}$$

of $M_{2\times 2}(\mathbb{R})$. Let $\vec{x} = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}$. We wish to find $[\vec{x}]_{\mathcal{B}}$. Consider the equation

$$a\begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} + b\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + c\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}.$$

To get the coordinate vector of $\vec{x}$ with respect to $\mathcal{B}$, we need to solve for a, b, c, d. Equating the entries of the matrices on the left and right hand side of the equals sign gives us the system of equations

$$\begin{aligned} 3a + b + c + d &= 1 \\ 2a + c + 4d &= -1 \\ 2a + b + c &= 0 \\ 2a + b + 3d &= 3. \end{aligned}$$

To solve this equation we create an augmented matrix and row reduce, giving

$$\left[\begin{array}{cccc|c} 3 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 4 & -1 \\ 2 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 3 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

Therefore

$$\vec{x} = 1\begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} + 1\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$$

and

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ -3 \\ 0 \end{bmatrix}.$$

Example 1.3.23 Earlier you may have noticed that there is some kind of similarity between $\mathbb{R}^3$ and $\mathcal{P}_2(\mathbb{R})$, and we can somehow identify the vectors

$$\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad \text{and} \quad \vec{w} = a + bx + cx^2.$$

Now we can get a glimpse as to how these two vectors may indeed be viewed as the same after picking bases for the two vector spaces. Consider the standard (ordered) bases

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad \mathcal{C} = \{1, x, x^2\}$$

for $\mathbb{R}^3$ and $\mathcal{P}_2(\mathbb{R})$ respectively. Then we see that

$$[\vec{v}]_{\mathcal{B}} = [\vec{w}]_{\mathcal{C}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Once we have chosen a basis for a vector space V, every vector can now be represented as a column vector. Column vectors, as we know, come with their own addition and scalar multiplication. A natural question to ask is whether or not the column vector addition and scalar multiplication agree with the addition and scalar multiplication on V. Since everything so far in this course has worked out so beautifully, it would be a huge surprise if this wasn't true! Indeed, it is true.

Theorem 1.3.24 Let V be a vector space over $\mathbb{F}$ with ordered basis $\mathcal{B}$. Then

$$[\vec{x}]_{\mathcal{B}} + [\vec{y}]_{\mathcal{B}} = [\vec{x} + \vec{y}]_{\mathcal{B}} \quad \text{and} \quad t[\vec{x}]_{\mathcal{B}} = [t\vec{x}]_{\mathcal{B}}$$

for all $\vec{x}, \vec{y} \in V$ and all $t \in \mathbb{F}$.

Proof: This is just a matter of using the definition to determine $[\vec{x}]_{\mathcal{B}}, [\vec{y}]_{\mathcal{B}}, [\vec{x} + \vec{y}]_{\mathcal{B}}$ and $[t\vec{x}]_{\mathcal{B}}$. We'll leave the details as an easy exercise. □

EXERCISE

Prove Theorem 1.3.24.

Chapter 2

Linear Transformations

2.1 Linear Transformations Between Abstract Vectors

So far in the course we have studied vector spaces in isolation. That is, we've started with a single vector space and studied it, without looking at how it compares to, or interacts with, other vector spaces. However, we have seen glimpses that there is something to be said about comparing vector spaces. For example, $\mathbb{R}^3$ and $\mathcal{P}_2(\mathbb{R})$ appear to be the same vector space in some sense, just wrapped up in a different package.

In mathematics in general, when we want to study how objects interact, we usually think about functions between them. However, when studying functions between two vector spaces, we don't want to just take any old function. We'd like to take into account that we're working with vector spaces which come with the additional structure of vector addition and scalar multiplication. Ideally, our functions should respect this structure. This leads us to the following (hopefully familiar) definition.

Definition 2.1.1 **Linear Transformation, Linear Map, Linearity**

If V and W are vector spaces over $\mathbb{F}$, a function $L\colon V \to W$ is called a **linear transformation** (or **linear map**) if it satisfies the **linearity** properties:

1. $L(\vec{x} + \vec{y}) = L(\vec{x}) + L(\vec{y})$, and
2. $L(t\vec{x}) = tL(\vec{x})$

for all $\vec{x}, \vec{y} \in V$, $t \in \mathbb{F}$.

Said another way, it doesn't matter if you add two vectors before or after applying the linear map, and the same with scalar multiplication.

Example 2.1.2

A simple but important linear map is the **identity map** (or **identity transformation**) $\mathrm{id}\colon V \to V$, which sends each vector to itself: $\mathrm{id}(\vec{x}) = \vec{x}$.

Example 2.1.3 Consider the map $L\colon \mathcal{P}_3(\mathbb{R}) \to \mathbb{R}$ defined by $L(p) = p(2)$. This is a linear map. In general, let $t \in \mathbb{F}$. Define the **evaluation map**

$$\mathrm{ev}_t : \mathcal{P}_n(\mathbb{F}) \to \mathbb{F}$$

by $\mathrm{ev}_t(p) = p(t)$. This is a linear map, and the proof of this claim is left as an exercise.

EXERCISE

Let $t \in \mathbb{F}$. Prove that the evaluation map $\mathrm{ev}_t\colon \mathcal{P}_n(\mathbb{F}) \to \mathbb{F}$ is a linear map.

Example 2.1.4 Let $\mathrm{tr}\colon M_{n\times n}(\mathbb{F}) \to \mathbb{F}$ be the map defined by taking the trace of a matrix. Recall, if

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

then $\mathrm{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$. We will prove that tr is a linear map.

Let $A, B \in M_{n\times n}(\mathbb{F})$ with $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}$. Then

$$\begin{aligned} \mathrm{tr}(A+B) &= a_{11} + b_{11} + a_{22} + b_{22} + \cdots + a_{nn} + b_{nn} \\ &= a_{11} + a_{22} + \cdots + a_{nn} + b_{11} + b_{22} + \cdots + b_{nn} \\ &= \mathrm{tr}(A) + \mathrm{tr}(B). \end{aligned}$$

If $t \in \mathbb{F}$, then

$$\begin{aligned} \mathrm{tr}(tA) &= ta_{11} + ta_{22} + \cdots + ta_{nn} \\ &= t(a_{11} + a_{22} + \cdots + a_{nn}) \\ &= t(\mathrm{tr}(A)) \end{aligned}$$

so tr is a linear map.

Several other natural operations you are familiar with are linear maps. For example, integration and differentiation of polynomials are both linear maps.

Example 2.1.5 The map

$$D\colon \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$$

given by $D(p) = p'(x)$, or more explicitly by

$$D(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + 2a_2x + 3a_3x^2,$$

can be shown to be a linear map.

Similarly, the map

$$I\colon \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_4(\mathbb{R})$$

given by $I(p) = \int_0^x p(t)\ dt$, or more explicitly by

$$I(a_0 + a_1x + a_2x^2 + a_3x^3) = a_0x + \frac{1}{2}a_1x^2 + \frac{1}{3}a_2x^3 + \frac{1}{4}a_3x^4,$$

is also a linear map.

EXERCISE

Show that the maps D and I in the previous example are linear maps.

2.2 Rank and Nullity

As we did for linear maps from $\mathbb{F}^n$ to $\mathbb{F}^m$, we associate two important subspaces to each linear map $L\colon V \to V$.

Definition 2.2.1 **Range, Kernel, Nullspace**

Let $L\colon V \to W$ be a linear map. The **range** of L is

$$\text{Range}(L) = \{L(\vec{x}) \in W : \vec{x} \in V\}.$$

The **kernel** (or **nullspace**) of L is

$$\text{Ker}(L) = \{\vec{x} \in V : L(\vec{x}) = \vec{0}\}.$$

The kernel of a linear map $L\colon V \to W$ is the set of all the vectors in V that are mapped to $\vec{0} \in W$. The range of L is all the vectors in W that are outputs of L.

Theorem 2.2.2

Let V and W be vector spaces over $\mathbb{F}$, and let $L\colon V \to W$ be a linear map. Then

(a) $L(\vec{0}) = \vec{0}$,

(b) $\text{Range}(L)$ is a subspace of W, and

(c) $\text{Ker}(L)$ is a subspace of V.

Proof: The same proof that you've seen for linear maps from $\mathbb{F}^n$ to $\mathbb{F}^m$ works here. So we will only give a proof of part (c) and leave parts (a) and (b) to you.

Let's use the Subspace Test. From (a), we see that $\vec{0} \in \text{Ker}(L)$, so $\text{Ker}(L)$ is non-empty. Suppose next that $\vec{v}, \vec{w} \in \text{Ker}(L)$. Then $L(\vec{v} + \vec{w}) = L(\vec{v}) + L(\vec{w}) = \vec{0} + \vec{0} = \vec{0}$ so $\vec{v} + \vec{w} \in \text{Ker}(L)$, and therefore $\text{Ker}(L)$ is closed under addition. Finally, let $t \in \mathbb{F}$. Then $L(t\vec{v}) = tL(\vec{v}) = \vec{0}$, so $\text{Ker}(L)$ is closed under scalar multiplication. Thus $\text{Ker}(L)$ is a subspace of V. $\square$

EXERCISE

Prove parts (a) and (b) of Theorem 2.2.2.

Example 2.2.3

Consider the linear map map $L\colon \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a \\ b \end{bmatrix}.$$

Then $\operatorname{Ker}(L) = \left\{ \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix} \in \mathbb{R}^3 : c \in \mathbb{R} \right\}$ and $\operatorname{Range}(L) = \mathbb{R}^2$.

Example 2.2.4

Let $L\colon M_{2\times 2}(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ be defined by

$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = b + c + (c-d)x^2.$$

We leave it to you to check that L is linear. We have

$$\operatorname{Ker}(L) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) : b + c = c - d = 0 \right\} = \left\{ \begin{bmatrix} a & -c \\ c & c \end{bmatrix} : a, c \in \mathbb{R} \right\}.$$

It is clear that $\operatorname{Range}(L) \subseteq \operatorname{Span}(\{1, x^2\})$. Since $L\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = 1$ and $L\left(\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}\right) = x^2$, we see $\operatorname{Range}(L) \supseteq \operatorname{Span}(\{1, x^2\})$. Therefore $\operatorname{Range}(L) = \operatorname{Span}(\{1, x^2\})$.

If you examine these examples, you'll notice something interesting about the dimensions of the vector spaces involved. In the first example, $\dim(\mathbb{R}^3) = 3$, $\dim(\operatorname{Range}(L)) = 2$ and $\dim(\operatorname{Ker}(L)) = 1$. In the second we have $\dim(M_{2\times 2}(\mathbb{R})) = 4$, $\dim(\operatorname{Ker}(L)) = 2$ and $\dim(\operatorname{Range}(L)) = 2$. The dimension of the domain of L in both cases seems to be equal to the sum of the dimensions of $\operatorname{Range}(L)$ and $\operatorname{Ker}(L)$. Something interesting is going on, so let's give these dimensions some names.

Definition 2.2.5
Rank, Nullity

Let V and W be vector spaces over $\mathbb{F}$. The **rank** of a linear map $L\colon V \to W$ is the dimension of the range of L. The **nullity** of L is the dimension of the kernel (nullspace) of L. That is,

$$\operatorname{rank}(L) = \dim(\operatorname{Range}(L)) \quad \text{and} \quad \operatorname{nullity}(L) = \dim(\operatorname{Ker}(L)).$$

The key result about the rank and nullity of a linear map is the following theorem.

Theorem 2.2.6

(Rank–Nullity Theorem)

Let V and W be vector spaces over $\mathbb{F}$ with $\dim(V) = n$. Let $L\colon V \to W$ be a linear map. Then $\operatorname{rank}(L) + \operatorname{nullity}(L) = n$.

The idea of the proof is as follows. We will start with a basis of $\text{Ker}(L)$ of size k and we will extend this to a basis of V by adding m vectors to it (so $\dim(V) = n + m$). Then we prove that the image of the m new vectors under L give a basis for $\text{Range}(W)$, which will complete the proof.

Proof: Let $\{\vec{v}_1, \ldots, \vec{v}_k\}$ be a basis for $\text{Ker}(L)$ so $\text{nullity}(L) = k$. Extend this to a basis $\{\vec{v}_1, \ldots, \vec{v}_k, \vec{w}_1, \ldots, \vec{w}_m\}$ for V so $\dim(V) = k + m$. It suffices to that show $\mathcal{B} = \{L(\vec{w}_1), \ldots, L(\vec{w}_m)\}$ is a basis for $\text{Range}(L)$. We first show $\text{Span}(\mathcal{B}) = \text{Range}(L)$. Clearly $\text{Span}(\mathcal{B}) \subseteq \text{Range}(L)$, so we must prove the reverse containment $\supseteq$. Let $\vec{w} \in \text{Range}(L)$. Then $\vec{w} = L(\vec{v})$ for some $\vec{v} \in V$, and we may write $\vec{v}$ as

$$\vec{v} = t_1\vec{v}_1 + \cdots + t_k\vec{v}_k + s_1\vec{w}_1 + \cdots + s_m\vec{w}_m.$$

Then

$$\begin{aligned}\vec{w} = L(\vec{v}) &= L(t_1\vec{v}_1 + \cdots + t_k\vec{v}_k + s_1\vec{w}_1 + \cdots + s_m\vec{w}_m)\\ &= t_1L(\vec{v}_1) + \cdots + t_kL(\vec{v}_k) + s_1L(\vec{w}_1) + \cdots + s_mL(\vec{w}_m)\\ &= s_1L(\vec{w}_1) + \cdots + s_mL(\vec{w}_m).\end{aligned}$$

So $\mathcal{B}$ is a spanning set for $\text{Range}(L)$. For linear independence, suppose that

$$s_1L(\vec{w}_1) + \cdots + s_mL(\vec{w}_m) = \vec{0}.$$

Since L is linear, this implies $s_1\vec{w}_1 + \cdots + s_m\vec{w}_m \in \text{Ker}(L)$. Therefore

$$s_1\vec{w}_1 + \cdots + s_m\vec{w}_m = t_1\vec{v}_1 + \cdots + t_k\vec{v}_k$$

for some $t_1, \ldots, t_k \in \mathbb{F}$. However, $\{\vec{v}_1, \ldots, \vec{v}_k, \vec{w}_1, \ldots, \vec{w}_m\}$ is linearly independent, so we must conclude $s_1 = \cdots = s_m = t_1 = \cdots = t_k = 0$. Therefore $\mathcal{B}$ is a basis for $\text{Range}(L)$ and so $\text{rank}(L) = m$. Since $\text{nullity}(L) = k$ and $\dim(V) = m + k$, the proof is complete. □

Here are some examples of the kinds of things you can conclude with the Rank–Nullity theorem in your back pocket.

Example 2.2.7 Let $L\colon \mathcal{P}_3(\mathbb{R}) \to \mathbb{R}^3$ be a linear map. Since $\dim(\mathbb{R}^3) = 3$, it must be that $\text{rank}(L) \leq 3$. Since $\dim(\mathcal{P}_3(\mathbb{R})) = 4$, the Rank–Nullity theorem implies $\text{nullity}(L) \geq 1$. Therefore without knowing anything about the linear map, we can conclude that there is at least one non-zero vector $\vec{v} \in \mathcal{P}_3(\mathbb{R})$ such that $L(\vec{v}) = \vec{0}$.

Example 2.2.8 Let $L\colon \mathbb{C}^4 \to M_{2\times2}(\mathbb{C})$ be a linear map. Then $\text{Ker}(L) = \{\vec{0}\}$ if and only if $\text{Range}(L) = M_{2\times2}(\mathbb{C})$.

Proof: First note $\dim(\mathbb{C}^4) = \dim(M_{2\times2}(\mathbb{C})) = 4$. If $\text{Ker}(L) = \{\vec{0}\}$ then $\text{nullity}(L) = 0$ so the Rank–Nullity theorem says $\text{rank}(L) = 4$. Therefore $\text{Range}(L)$ is a 4-dimensional subspace of $M_{2\times2}(\mathbb{C})$ so it must be that $\text{Range}(L) = M_{2\times2}(\mathbb{C})$ (why?). Conversely, if $\text{Range}(L) = M_{2\times2}(\mathbb{C})$, then $\text{rank}(L) = 4$. Therefore $\text{nullity}(L) = 0$ so $\text{Ker}(L) = \{\vec{0}\}$. □

2.3 Linear Maps as Matrices

Recall that given an $m \times n$ matrix in $A \in M_{m\times n}(\mathbb{F})$, we can define a linear map $L\colon \mathbb{F}^n \to \mathbb{F}^m$ by $L(\vec{x}) = A\vec{x}$. Conversely, you learned that every linear map $L\colon \mathbb{F}^n \to \mathbb{F}^m$ can be realized in this form for an appropriate matrix A. So, in some sense, linear maps between $\mathbb{F}^n$ and $\mathbb{F}^m$ and matrices in $M_{m\times n}(\mathbb{F})$ are two sides of the same coin.

Example 2.3.1 Consider the linear map $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a+2b \\ a-2b \end{bmatrix}$. We can find a matrix that performs this transformation. Indeed,

$$\begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a+2b \\ a-2b \end{bmatrix}.$$

The fact that a matrix even existed in the previous example was plausible because it's not much of a stretch of the imagination to view a vector in $\mathbb{R}^2$ as a column vector. But we can do this in *every vector space*—once we fix a basis! Recall that if we fix an ordered basis for a vector space, then every vector can be written as a column vector by simply taking its coordinate vector.

So, now that we have this, it's reasonable to ask whether or not every linear map can be viewed as a matrix transformation on coordinate vectors. Let's take a look at an example.

Example 2.3.2 Let $L\colon \mathcal{P}_2(\mathbb{F}) \to M_{2\times 2}(\mathbb{F})$ be the linear map defined by $L(a+bx+cx^2) = \begin{bmatrix} a-2b & 4c \\ a+b+c & b-c \end{bmatrix}$. Fix the standard (ordered) bases

$$\mathcal{B} = \{1, x, x^2\} \quad \text{and} \quad \mathcal{C} = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

for $\mathcal{P}_2(\mathbb{F})$ and $M_{2\times 2}(\mathbb{F})$, respectively. The coordinate vectors of $\vec{v} = a + bx + cx^2$ and $L(\vec{v}) = \begin{bmatrix} a-2b & 4c \\ a+b+c & b-c \end{bmatrix}$ are

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad \text{and} \quad [L(\vec{v})]_{\mathcal{C}} = \begin{bmatrix} a-2b \\ 4c \\ a+b+c \\ b-c \end{bmatrix}.$$

So, if there is a matrix A which performs the linear map for us (by matrix multiplication of course), it must be such that

$$A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a-2b \\ 4c \\ a+b+c \\ b-c \end{bmatrix}.$$

We first note that if A is to exist, it must be a 4×3 matrix. With that in mind, if we stare at this for a while (we'll explain how to do this more systematically later on) we can see that we can take

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 4 \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

For some foreshadowing of notation, we let ${}_{\mathcal{C}}[L]_{\mathcal{B}} = A$.

The previous example is not just a fortunate coincidence. We are always able to find a matrix that performs any given linear transformation, once we fix bases and work with the resulting coordinate vectors. This is the content of our next theorem.

Before we state and prove it though, it is worth addressing why we'd care to do this. Matrices come equipped with machinery to compute many things. It will turn out that once we turn our linear map into a matrix, we can use this machinery to learn about our linear map.

Theorem 2.3.3 Let V be an n-dimensional vector space with ordered basis $\mathcal{B}$. Let W be an m-dimensional vector space with ordered basis $\mathcal{C}$. Then, for every linear map $L\colon V \to W$, there exists an $m \times n$ matrix A such that $[L(\vec{v})]_{\mathcal{C}} = A[\vec{v}]_{\mathcal{B}}$ for all $\vec{v} \in V$. Conversely, every $m \times n$ matrix A defines a linear map $L\colon V \to W$ by $[L(\vec{v})]_{\mathcal{C}} = A[\vec{v}]_{\mathcal{B}}$.

Proof: Since matrix multiplication satisfies $A(B+C) = AB + AC$ and $t(AB) = A(tB)$ for all matrices A, B, C and all scalars $t \in \mathbb{F}$, A defines a linear map $L\colon V \to W$ by $A[\vec{v}]_{\mathcal{B}} = [L(\vec{v})]_{\mathcal{C}}$.

For the forward direction, let $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ and $\mathcal{C} = \{\vec{w}_1, \ldots, \vec{w}_m\}$. Let $\vec{v} \in V$, then $\vec{v} = t_1\vec{v}_1 + \cdots + t_n\vec{v}_n$ and $L(\vec{v}) = s_1\vec{w}_1 + \cdots + s_m\vec{w}_m$. Since L is linear we have

$$L(\vec{v}) = t_1 L(\vec{v}_1) + \cdots + t_n L(\vec{v}_n) = s_1\vec{w}_1 + \cdots + s_m\vec{w}_m.$$

For each $i \in \{1, \ldots, n\}$, let $L(\vec{v}_i) = a_{1i}\vec{w}_1 + \cdots + a_{mi}\vec{w}_m$. Then

$$\begin{aligned} L(\vec{v}) &= s_1\vec{w}_1 + \cdots + s_m\vec{w}_m \\ &= t_1(a_{11}\vec{w}_1 + \cdots + a_{m1}\vec{w}_m) + \cdots + t_n(a_{1n}\vec{w}_1 + \cdots a_{mn}\vec{w}_m) \\ &= (a_{11}t_1 + a_{12}t_2 + \cdots + a_{1n}t_n)\vec{w}_1 + \cdots + (a_{m1}t_1 + \cdots + a_{mn}t_n)\vec{w}_m. \end{aligned}$$

Therefore we have $s_i = a_{i1}t_1 + \cdots + a_{in}t_n$ for all $i \in \{1, \ldots, m\}$. This is of course how matrix multipication works, and we see

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix} = \begin{bmatrix} s_1 \\ \vdots \\ s_m \end{bmatrix}.$$

Since $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix}$ and $[L(\vec{v})]_{\mathcal{C}} = \begin{bmatrix} s_1 \\ \vdots \\ s_m \end{bmatrix}$, the proof is completed. □

Hidden in the proof is the fact that if $\vec{v}_i$ is the ith basis vector of $\mathcal{B}$, then $[L(\vec{v}_i)]_{\mathcal{C}}$ is simply the ith column of the desired matrix A. This gives us the following corollary.

Corollary 2.3.4

Let V be a vector space with ordered basis $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$. Let W be a vector space with ordered basis $\mathcal{C} = \{\vec{c}_1, \ldots, \vec{c}_m\}$. Let $L\colon V \to W$ be a linear map. Then the $m \times n$ matrix A such that $[L(\vec{v})]_{\mathcal{C}} = A[\vec{v}]_{\mathcal{B}}$ for all $\vec{v} \in V$, which we denote ${}_{\mathcal{C}}[L]_{\mathcal{B}}$, is given by

$$ {}_{\mathcal{C}}[L]_{\mathcal{B}} = \begin{bmatrix} [L(\vec{b}_1)]_{\mathcal{C}} & \cdots & [L(\vec{b}_n)]_{\mathcal{C}} \end{bmatrix}. $$

The fact that the matrix contains all the information of L, and is determined by the images of the basis vectors, tells us something very interesting about linear maps: They are entirely determined by where they send a basis.

The matrix A for a linear map L is determined once you pick ordered bases for each vector space. We will give this matrix a name.

Definition 2.3.5
Matrix of a Linear Map

We call the matrix ${}_{\mathcal{C}}[L]_{\mathcal{B}}$ the **matrix of the linear map** L with respect to the ordered bases $\mathcal{B}$ and $\mathcal{C}$. If $L\colon V \to V$ and we choose the same basis $\mathcal{B}$ for both the domain and codomain of L, then we will write $[L]_{\mathcal{B}} = {}_{\mathcal{B}}[L]_{\mathcal{B}}$.

Example 2.3.6

Consider the differentiation map $D\colon \mathcal{P}_3(\mathbb{C}) \to \mathcal{P}_2(\mathbb{C})$, and let both vector spaces be endowed with the standard ordered bases $\mathcal{B}$ and $\mathcal{C}$ respectively. Then $D(1) = 0$, $D(x) = 1$, $D(x^2) = 2x$, and $D(x^3) = 3x^2$. Therefore

$$ \begin{aligned} {}_{\mathcal{C}}[D]_{\mathcal{B}} &= \begin{bmatrix} [D(1)]_{\mathcal{C}} & [D(x)]_{\mathcal{C}} & [D(x^2)]_{\mathcal{C}} & [D(x^3)]_{\mathcal{C}} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}. \end{aligned} $$

Let's confirm that this works with a specific example. Let $\vec{v} = 4 + 2x + (-2)x^2 + ix^3$. Then $D(\vec{v}) = 2 - 4x + 3ix^2$ so $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 2 \\ -2 \\ i \end{bmatrix}$ and $[D(\vec{v})]_{\mathcal{C}} = \begin{bmatrix} 2 \\ -4 \\ 3i \end{bmatrix}$. Indeed, we can check that

$$ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ -2 \\ i \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 3i \end{bmatrix}. $$

If you dwell on Theorem 2.3.3, it becomes apparent that the theorem only works because of the way matrix multiplication is defined. In fact, the reason matrix multiplication is defined the way it is is essentially so that Theorem 2.3.3 is true! Even better, the next fact is also true, although we will not prove it here. If you feel like a moderately difficult challenge, you should prove it! You definitely have the tools to do so at this point in the course. The proof is more a matter of careful bookkeeping.

Proposition 2.3.7 Let V, U, and W be vector spaces over $\mathbb{F}$ with bases $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ respectively. Let $L\colon V \to U$ and $M : U \to W$ be linear maps. Then ${}_{\mathcal{D}}[M \circ L]_{\mathcal{B}} = {}_{\mathcal{D}}[M]_{\mathcal{C}}\ {}_{\mathcal{C}}[L]_{\mathcal{B}}$.

EXERCISE

Prove Proposition 2.3.7.

Now, let's see some of the computational power of matrices in action. First we recall the notions of a column space and nullspace for a matrix, and see how they relate to the range and nullspace of a linear map.

Definition 2.3.8 **Column Space, Nullspace of a Matrix**

Let $A \in M_{m\times n}(\mathbb{F})$.

The **column space** of A, denoted by $\operatorname{Col}(A)$, is the span of the columns of A.

The **nullspace** of A, denoted by $\operatorname{Null}(A)$, is the set of all $\vec{v} \in \mathbb{F}^n$ such that $A\vec{v} = \vec{0}$.

REMARK

Recall that if we let $L\colon \mathbb{F}^n \to \mathbb{F}^m$ be the linear transformation determined by $A \in M_{m\times n}(\mathbb{F})$, namely the one defined by $L(\vec{x}) = A\vec{x}$, then

$$\operatorname{Col}(A) = \operatorname{Range}(L) \quad \text{and} \quad \operatorname{Null}(A) = \operatorname{Ker}(L).$$

This shows, in particular, that $\operatorname{Col}(A)$ and $\operatorname{Null}(A)$ are subspaces of $\mathbb{F}^m$ and $\mathbb{F}^n$, respectively. We will see below that a similar type of result is true for linear mappings between general vector spaces.

Let's quickly review how we can find bases for the column space and nullspace of a given matrix. For the column space, one row-reduces the matrix and chooses the original columns corresponding to the the columns with leading ones in them. For the nullspace, one solves the system of equations given by augmenting the matrix with a column of 0 and then taking the basic solutions. Let's see this in an example.

Example 2.3.9 Find a basis for $\operatorname{Col}(A)$ and $\operatorname{Null}(A)$ where $A = \begin{bmatrix} 1 & 2 & 5 & -3 & -8 \\ -2 & -4 & -11 & 2 & 4 \\ -1 & -2 & -6 & -1 & -4 \\ 1 & 2 & 5 & -2 & -5 \end{bmatrix}$.

Solution:

First, we put the matrix A into row reduced echelon form, which is given by

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

So, immediately, we have that a basis for $\mathrm{Col}(A)$ is

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -11 \\ -6 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ -1 \\ -2 \end{bmatrix} \right\}.$$

Finding a basis for $\mathrm{Null}(A)$ is a little more involved. Finding a vector $\vec{v} = \begin{bmatrix} x_1 \\ \vdots \\ x_5 \end{bmatrix}$ such that $A\vec{v} = \vec{0}$ is the same as solving the system of equations given by the augmented matrix $[A \mid \mathbf{0}]$. The row-reduced augmented matrix is given by

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

If we let the variables for this system be $x_1, \ldots, x_5$, we can write down an entire set of solutions as follows. For every column not corresponding to a leading 1, we let that variable be a free variable, and solve for the rest of them. In this example, the free variables are x_2 and x_5, so let $x_2 = s$ and $x_5 = t$. Then

$$\begin{aligned} x_1 &= -t - 2s \\ x_2 &= s \\ x_3 &= 0 \\ x_4 &= -3t \\ x_5 &= t \end{aligned}$$

so every vector in $\mathrm{Null}(A)$ is of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 0 \\ -3 \\ 1 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Finally, we write down a basis for $\mathrm{Null}(A)$ as

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

The next proposition allows us to harness the computational power of matrices to learn about the range and kernel of a linear map.

Proposition 2.3.10 Let $L\colon V \to W$ be a linear map and $\mathcal{B}$ and $\mathcal{C}$ bases for V and W respectively. Let $A = {}_{\mathcal{C}}[L]_{\mathcal{B}}$.

(a) $\vec{v} \in \operatorname{Ker}(L)$ if and only if $[\vec{v}]_{\mathcal{B}} \in \operatorname{Null}(A)$.

(b) $\vec{w} \in \operatorname{Range}(L)$ if and only if $[\vec{w}]_{\mathcal{C}} \in \operatorname{Col}(A)$.

Proof: (a) Observe that $\vec{v} \in V$ will be in $\operatorname{Ker}(L)$ if and only if $L(\vec{v}) = \vec{0}$ which is the case if and only if $[L(\vec{v})]_{\mathcal{C}} = [\vec{0}]_{\mathcal{C}}$. According to the definition of $A = {}_{\mathcal{C}}[L]_{\mathcal{B}}$, this last condition is equivalent to $A[\vec{v}]_{\mathcal{B}} = \vec{0}$. Thus $\vec{v} \in \operatorname{Ker}(L)$ is equivalent to $[\vec{v}]_{\mathcal{B}} \in \operatorname{Null}(A)$.

(b) Exercise.

□

EXERCISE

Prove part (b) of Proposition 2.3.10.

This proposition tells us that if we want to find a basis for the kernel and range of a linear map, we just need to pick some bases, find the matrix associated to the linear map and find bases for the nullspace and column space of the matrix.

Example 2.3.11 Consider the linear map $L\colon \mathcal{P}_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ given by

$$L(a + bx + cx^2) = \begin{bmatrix} a+b+c & a-b+3c \\ 3a+b+5c & 0 \end{bmatrix}.$$

Find bases for $\operatorname{Ker}(L)$ and $\operatorname{Range}(L)$.

Solution:

Let $\mathcal{B}$ and $\mathcal{C}$ be the standard bases for $\mathcal{P}_2(\mathbb{R})$ and $M_{2\times 2}(\mathbb{R})$ respectively. Since $L(1) = \begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix}$, $L(x) = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$, and $L(x^2) = \begin{bmatrix} 1 & 3 \\ 5 & 0 \end{bmatrix}$ we have

$${}_{\mathcal{C}}[L]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 3 \\ 3 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}.$$

Call this matrix A. We will now find bases for $\operatorname{Col}(A)$ and $\operatorname{Null}(A)$, and then convert this information back to find bases for $\operatorname{Range}(L)$ and $\operatorname{Ker}(L)$. Row reducing A gives

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 3 \\ 3 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

With a little work we compute bases for $\mathrm{Col}(A)$ and $\mathrm{Null}(A)$ to be

$$\left\{\begin{bmatrix}1\\1\\3\\0\end{bmatrix},\begin{bmatrix}1\\-1\\1\\0\end{bmatrix}\right\} \quad \text{and} \quad \left\{\begin{bmatrix}-2\\1\\1\end{bmatrix}\right\},$$

respectively. Since these are coordinate vectors, we finally have that

$$\left\{\begin{bmatrix}1&1\\3&0\end{bmatrix},\begin{bmatrix}1&-1\\1&0\end{bmatrix}\right\} \quad \text{and} \quad \{-2+x+x^2\}$$

are bases for $\mathrm{Range}(L)$ and $\mathrm{Ker}(L)$, respectively.

2.4 Change of Coordinates

We may be faced with a situation where we want to switch bases for the same vector space, because a particular problem is computationally easier to solve in one of the bases. We do this all the time in physics when we choose a set of coordinates that is natural with respect to the problem at hand.

If we are given bases $\mathcal{B}$ and $\mathcal{C}$ of a vector space V, it would be handy if we could find a matrix that takes a coordinate vector with respect to $\mathcal{B}$ and transforms it into the coordinate vector with respect to $\mathcal{C}$. This can be achieved by simply finding the matrix of the identity map $\mathrm{id}\colon V \to V$.

Example 2.4.1 Let $\mathcal{S} = \{1, x, x^2\}$ be the standard basis for $\mathcal{P}_2(\mathbb{R})$ and $\mathcal{B} = \{1, 1+x, 1+x+x^2\}$ another basis. We would like a matrix A such that $A[\vec{v}]_{\mathcal{S}} = [\vec{v}]_{\mathcal{B}}$. To find A, consider the linear map $\mathrm{id}\colon \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ given by $\mathrm{id}(\vec{v}) = \vec{v}$ for all $\vec{v} \in V$. We will find ${}_{\mathcal{B}}[\mathrm{id}]_{\mathcal{S}}$. This should be our desired matrix since ${}_{\mathcal{B}}[\mathrm{id}]_{\mathcal{S}}[\vec{v}]_{\mathcal{S}} = [\mathrm{id}(\vec{v})]_{\mathcal{B}} = [\vec{v}]_{\mathcal{B}}$. We will call this matrix ${}_{\mathcal{B}}\mathcal{I}_{\mathcal{S}}$.

We have

$$[\mathrm{id}(1)]_{\mathcal{B}} = [1]_{\mathcal{B}} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad [\mathrm{id}(x)]_{\mathcal{B}} = [x]_{\mathcal{B}} = \begin{bmatrix}-1\\1\\0\end{bmatrix}, \quad \text{and} \quad [\mathrm{id}(x^2)]_{\mathcal{B}} = [x^2]_{\mathcal{B}} = \begin{bmatrix}0\\-1\\1\end{bmatrix}.$$

Therefore

$${}_{\mathcal{B}}\mathcal{I}_{\mathcal{S}} = \begin{bmatrix}1&-1&0\\0&1&-1\\0&0&1\end{bmatrix}.$$

If this matrix does what it should, then we should be able to use it to switch coordinates from $\mathcal{S}$ to $\mathcal{B}$. Let's check. Take $\vec{v} = 0(1) + 2(1+x) + (-1)(1+x+x^2) = 1 + x - x^2$. Then

$$[\vec{v}]_{\mathcal{S}} = \begin{bmatrix}1\\1\\-1\end{bmatrix} \quad \text{and} \quad [\vec{v}]_{\mathcal{B}} = \begin{bmatrix}0\\2\\-1\end{bmatrix}.$$

And indeed,

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$$

which shows that ${}_{\mathcal{B}}\mathcal{I}_{\mathcal{S}}[\vec{v}]_{\mathcal{S}} = [\vec{v}]_{\mathcal{B}}$.

Example 2.4.2

Continuing from the previous example, let's try to find the matrix that changes $\mathcal{B}$-coordinates to $\mathcal{S}$-coordinates. By the same reasoning as above, this matrix should be ${}_{\mathcal{S}}[\mathrm{id}]_{\mathcal{B}}$, and we'll denote it by ${}_{\mathcal{S}}\mathcal{I}_{\mathcal{B}}$. We have

$$[1]_{\mathcal{S}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [1+x]_{\mathcal{S}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad [1+x+x^2]_{\mathcal{S}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Therefore

$${}_{\mathcal{S}}\mathcal{I}_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

If we let $\vec{v} = 1 + x - x^2$ as in the preceding example, we find that ${}_{\mathcal{S}}\mathcal{I}_{\mathcal{B}}[\vec{v}]_{\mathcal{B}} = [\vec{v}]_{\mathcal{S}}$:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

So ${}_{\mathcal{S}}\mathcal{I}_{\mathcal{B}}$ behaves as expected.

Definition 2.4.3

Change of Coordinates Matrix

Let V be a finite dimensional vector space, and let $\mathcal{B}$ and $\mathcal{C}$ be two bases for V. The **change of coordinates matrix** ${}_{\mathcal{C}}\mathcal{I}_{\mathcal{B}}$ is the matrix ${}_{\mathcal{C}}[\mathrm{id}]_{\mathcal{B}}$, where $\mathrm{id}\colon V \to V$ is the identity map.

This name makes sense since $[\vec{v}]_{\mathcal{C}} = {}_{\mathcal{C}}\mathcal{I}_{\mathcal{B}}\,[\vec{v}]_{\mathcal{B}}$ for all $\vec{v} \in V$.

Let us address the following natural question: What is the relationship between ${}_{\mathcal{C}}\mathcal{I}_{\mathcal{B}}$ and ${}_{\mathcal{B}}\mathcal{I}_{\mathcal{C}}$? Notice that

$${}_{\mathcal{C}}\mathcal{I}_{\mathcal{B}}\;{}_{\mathcal{B}}\mathcal{I}_{\mathcal{C}}\;[\vec{v}]_{\mathcal{C}} = [\vec{v}]_{\mathcal{C}} \quad \text{and} \quad {}_{\mathcal{B}}\mathcal{I}_{\mathcal{C}}\;{}_{\mathcal{C}}\mathcal{I}_{\mathcal{B}}\;[\vec{v}]_{\mathcal{B}} = [\vec{v}]_{\mathcal{B}}$$

for all $\vec{v} \in V$. With this in mind you are able to write up a proof of the next proposition.

Proposition 2.4.4

Let V be a finite dimensional vector space with bases $\mathcal{B}$ and $\mathcal{C}$. Then ${}_{\mathcal{C}}\mathcal{I}_{\mathcal{B}} = ({}_{\mathcal{B}}\mathcal{I}_{\mathcal{C}})^{-1}$.

EXERCISE

Prove Proposition 2.4.4.

Example 2.4.5 Referring to Examples 2.4.1 and 2.4.2, where we found

$$ {}_{\mathcal{S}}\mathcal{I}_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad {}_{\mathcal{B}}\mathcal{I}_{\mathcal{S}} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, $$

we see that

$$ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

which shows that ${}_{\mathcal{S}}\mathcal{I}_{\mathcal{B}}$ and ${}_{\mathcal{B}}\mathcal{I}_{\mathcal{S}}$ are inverses.

To close off this section, let's consider the following scenario. Suppose we have a linear map $L\colon V \to W$ and suppose we choose bases $\mathcal{B}$ and $\mathcal{C}$ for V and W, respectively. This allows us to create the matrix ${}_{\mathcal{C}}[L]_{\mathcal{B}}$. If we choose different bases $\mathcal{B}'$ and $\mathcal{C}'$ for V and W, then we are able to create the matrix ${}_{\mathcal{C}'}[L]_{\mathcal{B}'}$. How are ${}_{\mathcal{C}}[L]_{\mathcal{B}}$ and ${}_{\mathcal{C}'}[L]_{\mathcal{B}'}$ related? The obvious guess is that we should have

$$ {}_{\mathcal{C}'}[L]_{\mathcal{B}'} = {}_{\mathcal{C}'}\mathcal{I}_{\mathcal{C}}\ {}_{\mathcal{C}}[L]_{\mathcal{B}}\ {}_{\mathcal{B}}\mathcal{I}_{\mathcal{B}'}, $$

since to apply the transformation ${}_{\mathcal{C}'}[L]_{\mathcal{B}'}$ we can begin by changing coordinates on the domain from $\mathcal{B}'$ to $\mathcal{B}$, apply ${}_{\mathcal{C}}[L]_{\mathcal{B}}$ and then change coordinates from $\mathcal{C}$ to $\mathcal{C}'$. This guess is indeed correct. It's also worth noticing that our notation works very well in this scenario.

Proposition 2.4.6 Let $L\colon V \to W$ be a linear map between two finite-dimensional vector spaces V and W. Suppose that $\mathcal{B}$ and $\mathcal{B}'$ are ordered bases for V and that $\mathcal{C}$ and $\mathcal{C}'$ are ordered bases for W. Then

$$ {}_{\mathcal{C}'}[L]_{\mathcal{B}'} = {}_{\mathcal{C}'}\mathcal{I}_{\mathcal{C}}\ {}_{\mathcal{C}}[L]_{\mathcal{B}}\ {}_{\mathcal{B}}\mathcal{I}_{\mathcal{B}'}. $$

Proof: This is just a matter of unpacking the definitions. We have

$$ \begin{aligned} {}_{\mathcal{C}'}\mathcal{I}_{\mathcal{C}}\ {}_{\mathcal{C}}[L]_{\mathcal{B}}\ {}_{\mathcal{B}}\mathcal{I}_{\mathcal{B}'}\ [\vec{v}]_{\mathcal{B}'} &= {}_{\mathcal{C}'}\mathcal{I}_{\mathcal{C}}\ {}_{\mathcal{C}}[L]_{\mathcal{B}}\ [\vec{v}]_{\mathcal{B}} \\ &= {}_{\mathcal{C}'}\mathcal{I}_{\mathcal{C}}\ [L(\vec{v})]_{\mathcal{C}} \\ &= [L(\vec{v})]_{\mathcal{C}'} \\ &= {}_{\mathcal{C}'}[L]_{\mathcal{B}'}[\vec{v}]_{\mathcal{B}'} \end{aligned} $$

for all $\vec{v} \in V$. This completes the proof (why?). □

Example 2.4.7 Let $\mathcal{S}$ and $\mathcal{B}$ be the bases for $\mathcal{P}_2(\mathbb{R})$ from Examples 2.4.1 and 2.4.2, where we had determined ${}_{\mathcal{B}}\mathcal{I}_{\mathcal{S}}$ and ${}_{\mathcal{S}}\mathcal{I}_{\mathcal{B}}$.

Consider the linear map $D\colon \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ given by differentiation. We'll leave it as an exercise for you to check that $[D]_{\mathcal{S}} = {}_{\mathcal{S}}[D]_{\mathcal{S}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. Let's find $[D]_{\mathcal{B}} = {}_{\mathcal{B}}[D]_{\mathcal{B}}$. We can do so either directly from the definition or by using Proposition 2.4.6.

For the direct approach, we simply note that $[D(1)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $[D(1+x)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and $[D(1+x+x^2)]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$. Therefore

$$[D]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

On the other hand, using Proposition 2.4.6, we get

$$[D]_{\mathcal{B}} = {}_{\mathcal{B}}\mathcal{I}_{\mathcal{S}}\ [D]_{\mathcal{S}}\ {}_{\mathcal{S}}\mathcal{I}_{\mathcal{B}} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

as expected!

2.5 Isomorphisms of Vector Spaces

Let's return to an observation that has come up a few times so far: $\mathbb{R}^3$ and $\mathcal{P}_2(\mathbb{R})$ are essentially the same! We could just rename the element $\begin{bmatrix} a & b & c \end{bmatrix}^T$ to $a + bx + cx^2$ and everything would work exactly the same. Somehow it feels like these two elements are the same thing called by different names. We will soon see that the vector spaces $\mathbb{R}^3$ and $\mathcal{P}_2(\mathbb{R})$, while technically different objects, have exactly the same structure. More precisely, we will see that $\mathbb{R}^3$ and $\mathcal{P}_2(\mathbb{R})$ are *isomorphic*.

An isomorphism between vector spaces should be thought of like a translator. It's a linear map that preserves information perfectly. No information is lost, and no information is missed. Let's look at a couple of examples to get a little more intuition.

Example 2.5.1 Consider the linear map $L\colon \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}^2$ given by $L(p) = \begin{bmatrix} p(0) \\ p(0) \end{bmatrix}$. This linear map is not an isomorphism because somehow it loses information. For example, $L(x+2) = L(x^2+2) = L(2) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ so just by looking at the output of L, we can't tell the difference between $x+2$ and 2 for example. Furthermore, L somehow misses information. For example, nothing maps to the vector $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

Example 2.5.2 On the other hand, consider the linear map $L\colon \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}^3$ given by

$$L(p) = \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix}.$$

There are two very interesting things about this map. First, it turns out that if you know the value of a polynomial in $\mathcal{P}_2(\mathbb{R})$ evaluated at three distinct points, you are able to recover the polynomial. That is, if $L(p) = L(q)$ then $p = q$. Furthermore, for any three numbers $a, b, c \in \mathbb{R}$, there is a polynomial $p \in \mathcal{P}_2(\mathbb{R})$ such that $p(-1) = a$, $p(0) = b$, and $p(1) = c$. Therefore, $\text{Range}(L) = \mathbb{R}^3$. With these two pieces of information, we can see that L is a perfect dictionary between $\mathcal{P}_2(\mathbb{R})$ and $\mathbb{R}^3$ and both vector spaces contain the same information, just wrapped up in a different package.

Roughly, an isomorphism of vector spaces will be a linear map $L\colon V \to W$ which is a perfect dictionary between V and W, that is, no information is lost, and no information is missed, after applying L. More formally, it will be a linear map that is injective and surjective, which we will now define.

Definition 2.5.3

Injective (One-to-One), Surjective (Onto)

Let $L\colon V \to W$ be a linear map between vector spaces.

We say L is **injective** (or **one-to-one**) if $L(\vec{v}_1) = L(\vec{v}_2)$ implies $\vec{v}_1 = \vec{v}_2$.

We say L is **surjective** (or **onto**) if $\text{Range}(L) = W$.

Here is a little result that will make checking injectivity that much easier.

Lemma 2.5.4

A linear map $L\colon V \to W$ is injective if and only if $\text{Ker}(L) = \{\vec{0}\}$.

Proof: Suppose $L\colon V \to W$ is injective and let $\vec{v} \in \text{Ker}(L)$. Then $L(\vec{v}) = L(\vec{0}) = \vec{0}$ so $\vec{v} = \vec{0}$. Therefore $\text{Ker}(L) \subseteq \{\vec{0}\}$, and since the reverse containment is obvious, it follows that $\text{Ker}(L) = \{\vec{0}\}$. Conversely, suppose $\text{Ker}(L) = \{0\}$ and let $L(\vec{v}) = L(\vec{w})$. Then $\vec{0} = L(\vec{v}) - L(\vec{w}) = L(\vec{v} - \vec{w})$ so $\vec{v} - \vec{w} \in \text{Ker}(L)$. Since the only vector in $\text{Ker}(L)$ is $\vec{0}$, we have $\vec{v} - \vec{w} = \vec{0}$ so $\vec{v} = \vec{w}$ completing the proof. $\square$

We now have all the pieces lined up for the following useful result.

Proposition 2.5.5

Let $L\colon V \to W$ be a linear map between finite-dimensional vector spaces. Then:

(a) L is injective if and only if $\text{nullity}(L) = 0$.

(b) L is surjective if and only if $\text{rank}(L) = \dim W$.

EXERCISE

Prove Proposition 2.5.5.

Example 2.5.6

Consider the linear map $L : \mathcal{P}_3(\mathbb{R}) \to \mathbb{R}^4$ given by

$$L(p(x)) = \begin{bmatrix} p(0) \\ p'(0) \\ p''(0) \\ p'''(0) \end{bmatrix},$$

where $p'(0)$, $p''(0)$, and $p'''(0)$ are the first, second, and third derivatives of p, evaluated at 0, respectively.

Now suppose $p(x) = a + bx + cx^2 + dx^3 \in \text{Ker}(L)$. Then $p(0) = p'(0) = p''(0) = p'''(0) = 0$ which yield the equations $a = 0$, $b = 0$, $2c = 0$, and $6d = 0$. We must conclude that $p(x) = 0$. Therefore the only polynomial in $\text{Ker}(L)$ is the zero polynomial, so $\text{nullity}(L) = 0$. By Proposition 2.5.5, we can immediately conclude that L is injective.

Since $\mathcal{P}_3(\mathbb{R})$ is a 4-dimensional vector space, the Rank–Nullity theorem tells us $\text{rank}(L) = 4$. Since $\dim(\mathbb{R}^4) = 4$, we can again exploit Proposition 2.5.5 to conclude L is surjective.

The injectivity and surjectivity of L in this example is telling us something very interesting about polynomials of degree at most 3. The injectivity says that such a polynomial p is entirely determined by the 4 numbers $p(0)$, $p'(0)$, $p''(0)$, and $p'''(0)$. The surjectivity of L says that given any 4 real numbers, you can find a polynomial p of degree at most 3 so that $p(0)$, $p'(0)$, $p''(0)$, and $p'''(0)$ are the desired 4 real numbers.

REMARK

If we choose bases $\mathcal{B}$ and $\mathcal{C}$ for V and W, then recall that we've shown that for all $\vec{v} \in V$,

$$\vec{v} \in \text{Ker}(L) \text{ if and only if } [\vec{v}]_{\mathcal{B}} \in \text{Null}({}_{\mathcal{C}}[L]_{\mathcal{B}}).$$

Since we can easily compute the nullspace of a matrix, it follows that we can easily check if a linear map is or is not injective.

Likewise, since for all $\vec{w} \in W$, $\vec{w} \in \text{Range}(L)$ if and only if $[\vec{w}]_{\mathcal{C}} \in \text{Col}({}_{\mathcal{C}}[L]_{\mathcal{B}})$, and we know how to compute the latter, we can easily determine if L is surjective.

Definition 2.5.7

Isomorphism, Isomorphic

Let $L\colon V \to W$ be a linear map. If L is injective and surjective, we say L is an **isomorphism**.

We say that two vector spaces V and W are **isomorphic**, and write $V \cong W$, if there is an isomorphism $L\colon V \to W$.

Let's see some examples.

Example 2.5.8

Consider $L\colon M_{2\times 2}(\mathbb{C}) \to \mathbb{C}^3$ given by $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+b \\ b-2c \\ a+b+d \end{bmatrix}$. Then $\begin{bmatrix} 2 & -2 \\ -1 & 0 \end{bmatrix} \in \text{Ker}(L)$ so L is not injective, and is therefore not an isomorphism.

Example 2.5.9 Let $L\colon \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}^3$ be the linear map given by $L(a+bx+cx^2) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. You should check that $\operatorname{Ker}(L) = \{\vec{0}\}$. The Rank–Nullity theorem now implies that $\operatorname{rank}(L) = 3$, implying $\operatorname{Range}(L)$ is a 3-dimensional subspace of $\mathbb{R}^3$, so $\operatorname{Range}(L) = \mathbb{R}^3$. Thus, L is an isomorphism and $\mathcal{P}_2(\mathbb{R})$ is isomorphic to $\mathbb{R}^3$ (so we can write $\mathcal{P}_2(\mathbb{R}) \cong \mathbb{R}^3$).

There can be more than one isomorphism between isomorphic vector spaces.

Example 2.5.10 Consider again the linear map $L\colon \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}^3$ given by

$$L(p) = \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix}.$$

Let's prove it is indeed an isomorphism, without assuming we know the fact that every degree at most 2 polynomial is uniquely determined by its values at 3 distinct points. Let's compute the nullspace and range of L by finding the matrix of the linear map with respect to the standard bases. Let $\mathcal{B}$ be the standard basis for $\mathcal{P}_2(\mathbb{R})$, and $\mathcal{C}$ the standard basis for $\mathbb{R}^3$. Since $L(1) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $L(x) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and $L(x^2) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Therefore

$$_{\mathcal{C}}[L]_{\mathcal{B}} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

which has row reduced echelon form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Since the identity matrix has 3 leading ones, $\operatorname{rank}(L) = 3$ so L is surjective. Applying the Rank–Nullity theorem gives $\operatorname{nullity}(L) = 0$. Therefore L is surjective and injective, and L is another isomorphism between $\mathcal{P}_2(\mathbb{R})$ and $\mathbb{R}^3$.

Suppose $V \cong W$. This does not imply that every linear map $L\colon V \to W$ is an isomorphism! Consider, for example, the linear map $L\colon \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}^3$ given by $L(p) = \vec{0}$ for all $p \in \mathcal{P}_2(\mathbb{R})$. Then $\operatorname{nullity}(L) = 3$ so L is not injective, even though $\mathcal{P}_2(\mathbb{R}) \cong \mathbb{R}^3$. Isomoprhisms are rather special linear maps.

If the intuition that an isomorphism is a kind of translator, then there should be a way to do an isomorphism in reverse, just like you should be able to translate a word back into English, if you had already translated it into Spanish. This is indeed true, and the next proposition makes it precise.

Proposition 2.5.11 A linear map $L\colon V \to W$ is an isomorphism if and only if there exists a *unique* linear map $L^{-1}\colon W \to V$ such that $L \circ L^{-1}(\vec{w}) = \vec{w}$ for all $\vec{w} \in W$ and $L^{-1} \circ L(\vec{v}) = \vec{v}$ for all $\vec{v} \in V$. In this case we call L^{-1} the **inverse linear map** to L.

Proof sketch: Given an isomorphism $L\colon V \to W$, define $L^{-1} : W \to V$ by $L^{-1}(\vec{w}) = \vec{v}_{\vec{w}}$ where $\vec{v}_{\vec{w}} \in V$ is the unique vector such that $L(\vec{v}_{\vec{w}}) = \vec{w}$. Such a unique vector exists since L is injective and surjective. It is left to you to prove that L^{-1} is a linear map satisfying the desired properties. For the converse direction, you should check that if such an inverse map exists, then L must necessarily be surjective and injective.

For uniqueness, suppose that $T\colon W \to V$ also satisfies the properties $L \circ T(\vec{w}) = \vec{w}$ for all $\vec{w} \in W$ and $T \circ L(\vec{v}) = \vec{v}$ for all $\vec{v} \in V$. Then $T(\vec{w}) = T(L \circ L^{-1}(\vec{w})) = (T \circ L)(L^{-1}(\vec{w})) = L^{-1}(\vec{w})$ for all $\vec{w} \in W$. So $T = L^{-1}$. □

EXERCISE

Fill in the details and complete the proof of Proposition 2.5.11.

Given an isomorphism, it is sometimes very easy to write down the inverse linear map, and sometimes not. For example, return to the isomorphism $L\colon \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}^3$ given by $L(a + bx + cx^2) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Then $L^{-1} : \mathbb{R}^3 \to \mathcal{P}_2(\mathbb{R})$ is given by $L^{-1}\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = a + bx + cx^2$. Let's check this is indeed the inverse. We have

$$L \circ L^{-1}\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = L(a + bx + cx^2) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

and

$$L^{-1} \circ L(a + bx + cx^2) = L^{-1}\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = a + bx + cx^2$$

so this is the inverse.

On the other hand, what is the inverse to the isomorphism $L\colon \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}^3$ given by $L(p) = \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix}$? The next proposition, which is a consequence of Proposition 2.5.11, gives us a way to find inverses to isomorphisms.

Proposition 2.5.12 Let $L\colon V \to W$ be an isomorphism. Let $\mathcal{B}$ be a basis for V, and $\mathcal{C}$ a basis for W. Then ${}_{\mathcal{C}}[L]_{\mathcal{B}}$ is an invertible matrix and $({}_{\mathcal{C}}[L]_{\mathcal{B}})^{-1} = {}_{\mathcal{B}}[L^{-1}]_{\mathcal{C}}$.

Proof: We have $L \circ L^{-1} = \mathrm{id}$, where $\mathrm{id}\colon W \to W$ is the identity map on W. Therefore,

$${}_{\mathcal{C}}[L \circ L^{-1}]_{\mathcal{C}} = {}_{\mathcal{C}}[\mathrm{id}]_{\mathcal{C}} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

On the other hand, ${}_{\mathcal{C}}[L \circ L^{-1}]_{\mathcal{C}} = {}_{\mathcal{C}}[L]_{\mathcal{B}}\ {}_{\mathcal{B}}[L^{-1}]_{\mathcal{C}}$ by Proposition 2.3.7. So we see that ${}_{\mathcal{C}}[L]_{\mathcal{B}}\ {}_{\mathcal{B}}[L^{-1}]_{\mathcal{C}}$ is equal to the identity matrix, hence ${}_{\mathcal{C}}[L]_{\mathcal{B}}^{-1} = {}_{\mathcal{B}}[L^{-1}]_{\mathcal{C}}$. □

Let's see this in action!

Example 2.5.13 Let $L\colon \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}^3$ be the isomorphism given by $L(p) = \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix}$. We have already seen that

$$_{\mathcal{C}}[L]_{\mathcal{B}} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Use your favourite method of computing the inverse of a matrix to show that

$$_{\mathcal{B}}[L^{-1}]_{\mathcal{C}} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}.$$

Now, since

$$\begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b \\ -\frac{1}{2}a + \frac{1}{2}c \\ \frac{1}{2}a - b + \frac{1}{2}c \end{bmatrix}$$

we have

$$L^{-1}\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = b + \left(-\frac{1}{2}a + \frac{1}{2}c\right)x + \left(\frac{1}{2}a - b + \frac{1}{2}c\right)x^2.$$

If we are to think of an isomorphism as simply a renaming of vectors, which we should, then we should expect two isomorphic vector spaces to have the same structure. At the very least, it wouldn't be unreasonable to expect two isomorphic vector spaces to have the same dimension. In fact, suppose $L\colon V \to W$ is a linear map. If $\dim(V) < \dim(W)$, then the Rank–Nullity theorem says $\mathrm{Range}(L)$ cannot be all of W, so L cannot be surjective. If $\dim(V) > \dim(W)$, the Rank–Nullity theorem says $\mathrm{nullity}(L) \geq 1$, so L cannot be injective. So if $V \cong W$ we at least must have that $\dim(V) = \dim(W)$. The natural thing to figure out now is whether or not we can have vector spaces (over the same field $\mathbb{F}$) of the same dimension that are not isomorphic. Before addressing this, let's formally record the two results that we just obtained, since they can be quite useful.

Proposition 2.5.14 Let $L\colon V \to W$ be a linear map between two finite-dimensional vector spaces. Then:

(a) If $\dim(V) < \dim(W)$, then L cannot be surjective.

(b) If $\dim(V) > \dim(W)$, then L cannot be injective.

EXERCISE

Give a careful proof of Proposition 2.5.14 by filling in the details given in the paragraph preceding it.

Now, suppose V and W have the same dimension, and pick bases for both. Then the coordinate vectors for both vector spaces are of the same type: they are column vectors with $\dim(V) = \dim(W)$ entries. This perhaps suggests that if $\dim(V) = \dim(W)$, then $V \cong W$, and indeed this is the case!

Theorem 2.5.15 Suppose V and W are finite dimensional vector spaces over the same field $\mathbb{F}$. Then V and W are isomorphic if and only if $\dim(V) = \dim(W)$.

Proof: Suppose $V \cong W$ via an isomorphism $L\colon V \to W$. Since L is injective, $\text{nullity}(L) = 0$ so the Rank–Nullity theorem implies $\dim(V) = \text{rank}(L)$. Since L is surjective, $\text{rank}(L) = \dim(W)$ so $\dim(V) = \dim(W)$.

Conversely, let $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis for V and $\mathcal{C} = \{\vec{w}_1, \ldots, \vec{w}_n\}$ a basis for W. Define a map $L\colon V \to W$ by

$$L(t_1\vec{v}_1 + \cdots + t_n\vec{v}_n) = t_1\vec{w}_1 + \cdots + t_n\vec{w}_n.$$

Note that L is linear since

$$\begin{aligned} L\left(\sum_{i=1}^n t_i\vec{v}_i + \sum_{i=1}^n s_i\vec{v}_i\right) &= L\left(\sum_{i=1}^n (t_i + s_i)\vec{v}_i\right) \\ &= \sum_{i=1}^n (t_i + s_i)\vec{w}_i \\ &= \sum_{i=1}^n t_i\vec{w}_i + \sum_{i=1}^n s_i\vec{w}_i \\ &= L\left(\sum_{i=1}^n t_i\vec{v}_i\right) + L\left(\sum_{i=1}^n s_i\vec{v}_i\right) \end{aligned}$$

and

$$\begin{aligned} L\left(\alpha\sum_{i=1}^n t_i\vec{v}_i\right) &= L\left(\sum_{i=1}^n \alpha t_i\vec{v}_i\right) \\ &= \sum_{i=1}^n \alpha t_i\vec{w}_i \\ &= \alpha\sum_{i=1}^n \alpha t_i\vec{w}_i \\ &= \alpha L\left(\sum_{i=1}^n t_i\vec{v}_i\right). \end{aligned}$$

To see that L is injective, suppose that $L(t_1\vec{v}_1 + \cdots + t_n\vec{v}_n) = t_1\vec{w}_1 + \cdots + t_n\vec{w}_n = \vec{0}$. Then since $\{\vec{w}_1, \ldots, \vec{w}_n\}$ is linearly independent, we must have $t_1 = \cdots = t_n = 0$ so $t_1\vec{v}_1 + \cdots + t_n\vec{v}_n = \vec{0}$ and so $\text{Ker}(L) = \{\vec{0}\}$. Finally, the Rank–Nullity theorem implies $\text{rank}(L) = \dim(V) = \dim(W)$ so L is surjective and is therefore an isomorphism. □

This is an incredibly powerful theorem. We immediately know that any two 7-dimensional vector spaces over $\mathbb{C}$, for example, are isomorphic. Furthermore, to find an isomorphism, we simply have to choose bases for both vector spaces and the map that appears in the proof of the theorem will be an isomorphism.

Corollary 2.5.16 Let V be an n-dimensioal vector space over $\mathbb{F}$. Then $V \cong \mathbb{F}^n$.

In fact, an isomorphism is given by the coordinate map $[\ ]_{\mathcal{B}} \colon V \to \mathbb{F}^n$, where $\mathcal{B}$ is any ordered basis for $\mathbb{F}$. (If you don't immediately see why, you should write down a careful proof!)

Chapter 3

Diagonalizability

3.1 Eigenvectors and Diagonalization

As hinted to before, sometimes the standard basis is not the best basis with which to study a particular problem, or a linear map. For example, consider the linear map $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \frac{2(x+y+z)}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

If $\mathcal{S}$ is the standard basis for $\mathbb{R}^3$, then you can check that

$$[L]_{\mathcal{S}} = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}.$$

Looking at this matrix, it's not clear what this linear map is doing, geometrically or otherwise. However, if we look at the basis

$$\mathcal{B} = \left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right\},$$

then it can be checked that

$$[L]_{\mathcal{B}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The effect of this matrix is easier to understand. It is a reflection! It negates the $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ direction and keeps the 2-dimensional subspace spanned by $\begin{bmatrix} -1 & 0 & 1 \end{bmatrix}^T$ and $\begin{bmatrix} -1 & 1 & 0 \end{bmatrix}^T$ unchanged.

This example shows us that sometimes looking at a particular problem with the right set of coordinates can prove enlightening. So, with this in mind, the following natural question arises: Given a linear map from a vector space to itself, how can we find an "enlightening" basis with which to view the linear map?

In this chapter we will restrict our attention to linear maps from a vector space to itself—and not between two different vector spaces. This is a fairly natural starting point, and we'll see that it leads to a reasonably nice theory.

Definition 3.1.1 **Linear Operator**

A linear map $T\colon V \to W$ is called a **linear operator** if $V = W$.

It would be nice to find vectors that are not rotated, but simply scaled when the linear map is applied to them. That is, we'd like to find vectors $\vec{v}$ such that $L(\vec{v}) = \lambda\vec{v}$ for some $\lambda \in \mathbb{F}$. If we can find a basis $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ of V such that $L(\vec{v}_i) = \lambda_i \vec{v}_i$ for every i, then with respect to $\mathcal{B}$ we would have

$$[L]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}.$$

Unfortunately, as well will see, we cannot always find such a basis. Let's try anyway!

What we're attempting to do should ring some bells: we are trying to *diagonalize* the linear operator L. The next definitions should come as no surprise.

Definition 3.1.2 **Eigenvector, Eigenvalue**

Let $L\colon V \to V$ be a linear operator. A non-zero vector $\vec{v} \in V$ such that $L(\vec{v}) = \lambda\vec{v}$ for some $\lambda \in \mathbb{F}$ is called an **eigenvector** of L. The number λ is called an **eigenvalue** of L.

Definition 3.1.3 **Eigenspace**

Let $L\colon V \to V$ be a linear operator, and let λ be an eigenvalue of L. The **eigenspace of L corresponding to λ** is

$$E_\lambda = \{\vec{v} \in V : L(\vec{v}) = \lambda\vec{v}\}.$$

These are the same definitions you've seen for square matrices. Since square matrices are essentially linear operators, as we've learned in Section 2.3, we will be able to transport all our results concerning eigenvalues, eigenvectors and the problem of diagonalizability from the setting of matrices to the setting of linear operators. For instance, we have the following.

Proposition 3.1.4

Let $L\colon V \to V$ be a linear operator, and let λ be an eigenvalue of L. The eigenspace of L corresponding to λ is a subspace of V.

EXERCISE

Prove Proposition 3.1.4.

The proof of this result is completely identical to the analogous proof about eigenspaces of matrices. We are going to spend the remainder of this chapter reviewing some of the theory of diagonalization of matrices, but rephrased in the language of linear operators. The proofs will be almost word-for-word identical to the matrix proofs, so we will leave them as exercises for the interested and particularly motivated reader!

To help make this new language a bit more familiar, let's look at a few examples.

Example 3.1.5

Let $D\colon \mathcal{P}_4(\mathbb{R}) \to \mathcal{P}_4(\mathbb{R})$ be the differentiation map. Then 0 is an eigenvalue of D since $D(3) = 0 = 0(3)$, and 3 is not the zero vector in $\mathcal{P}_4(\mathbb{R})$. Furthermore, 0 is the only eigenvalue of D. You can see this by noticing that λp and p have the same degree if and only if $\lambda \neq 0$. So, since $D(p)$ and p never have the same degree (unless $p = 0$ of course), then the only way $D(p) = \lambda p$ can be true is if $\lambda = 0$.

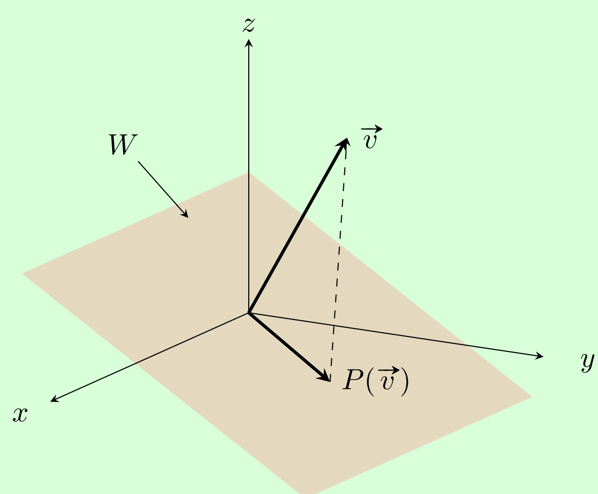
Now let's work out what the eigenspace E_0 corresponding to 0 looks like. By definition,

$$E_0 = \{p \in \mathcal{P}_4(\mathbb{R}) : D(p) = 0\}$$

and therefore $E_0 = \{p \in \mathcal{P}_4(\mathbb{R}) : p = k \text{ for some constant } k \in \mathbb{R}\}$ is the subspace of constant polynomials.

Example 3.1.6

Let W be a plane through the origin in $\mathbb{R}^3$ and let $P\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the projection onto W. That is, $P(\vec{v})$ is the vector projection of $\vec{v} \in \mathbb{R}^3$ onto W. We will study projections in more detail later in the course, so if they are not familiar to you, you can just use your intuition for now. In particular, convince yourself that P is linear.



What are the eigenvalues of P? There's an obvious one: $\lambda = 1$. Indeed, if $\vec{v} \in \mathbb{R}^3$ is in W, then $P(\vec{v}) = \vec{v} = 1\vec{v}$. This shows that all the vectors in W are eigenvectors with eigenvalue 1. In fact, the eigenspace corresponding to 1 is precisely W itself. That is, $E_1 = W$.

There is another geometrically obvious eigenvalue, namely $\lambda = 0$. In the next exercise you are asked to determine its eigenvectors.

EXERCISE

Let $P\colon \mathbb{R}^3 \to \mathbb{R}^3$ be as in the previous example.

(a) Complete the proof that $E_1 = W$ by showing that $E_1 \subseteq W$.

(b) Determine E_0.

It's all well and good to make definitions like the above, and work out examples where it's easy to find eigenvalues and eigenspaces by inspection, but how can we actually find eigenvalues and eigenspaces in a systematic way? As is becoming a pattern, we will pick a basis $\mathcal{B}$ of V, turn our linear operator into the matrix $[L]_{\mathcal{B}}$, and harness the computational power of matrices!

Once we've picked a basis, we can think of these definitions purely as definitions for matrices. In this case, we can think of a square matrix as a linear map from $\mathbb{F}^n$ to itself, and our vectors are column vectors in $\mathbb{F}^n$.

3.1.1 Finding Eigenvectors and Eigenvalues

To find eigenvectors and eigenvalues for a linear operator $L\colon V \to V$ on a finite-dimensional vector space V, first pick a basis $\mathcal{B}$ for V so you have an $n \times n$ matrix $A = [L]_{\mathcal{B}}$. Now the problem becomes finding eigenvalues and eigenvectors for A.

EXERCISE

Prove that $\vec{v} \in V$ is an eigenvector of L with eigenvalue λ if and only if $[\vec{v}]_{\mathcal{B}}$ is an eigenvector of A with eigenvalue λ.

To find an eigenvector for A, we're looking for a vector $\vec{v} \neq \vec{0}$ such that

$$A\vec{v} = \lambda\vec{v}$$

for some $\lambda \in \mathbb{F}$. If we rearrange this equation we get

$$A\vec{v} - \lambda\vec{v} = \vec{0}.$$

It would be tempting now to factor out the $\vec{v}$, which we will do, but we cannot as written. If we did, we would be left with a term $A - \lambda$, which makes no sense since A is a square matrix and λ is an element of $\mathbb{F}$. To get around this, we observe that $\lambda\vec{v} = \lambda I\vec{v}$ where I is the identity matrix of the appropriate size. Now our equation takes the form

$$(A - \lambda I)\vec{v} = \vec{0}.$$

If the matrix $A - \lambda I$ were invertible, then we could multiply both sides on the left by the inverse and get $\vec{v} = \vec{0}$. Since we're looking for non-zero vectors $\vec{v}$, this means we are looking for values of λ that make the matrix $A - \lambda I$ not invertible. Equivalently, we want values of λ such that $\det(A - \lambda I) = 0$. Furthermore, once we've found such a λ, a corresponding eigenvector is any non-zero vector such that $(A-\lambda I)\vec{v} = \vec{0}$, which must exist because $A - \lambda I$ is non-invertible. These are precisely the non-zero vectors in $\mathrm{Null}(A - \lambda I)$. Let's summarize.

Proposition 3.1.7 Let $L\colon V \to V$ be a linear operator on a finite-dimensional vector space V, and let λ be an eigenvalue of L. If $\mathcal{B}$ is an ordered basis for V and if $A = [L]_{\mathcal{B}}$, then the eigenspace of A corresponding to λ is $\mathrm{Null}(A - \lambda I)$:

$$E_\lambda = \mathrm{Null}(A - \lambda I).$$

Example 3.1.8

Let $D\colon \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ be the differentiation map. If $\mathcal{S} = \{1, x, x^2\}$ is the standard basis for $\mathcal{P}_2(\mathbb{R})$ then as we've noted before

$$[D]_{\mathcal{S}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Call this matrix A. The argument in Example 3.1.5 shows that 0 is an eigenvalue of D. To find the corresponding eigenspace, we must compute

$$E_0 = \operatorname{Null}(A - 0I) = \operatorname{Null}\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}\right).$$

To find the nullspace of a matrix, we simply apply the Gauss–Jordan algorithm and row reduce. Skipping the easy details, we find that

$$E_0 = \operatorname{Null}\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}\right) = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}.$$

Translating back to polynomials, the vector $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ is the polynomial $1 + 0x + 0x^2 = 1$. So we conclude that E_0 for the linear operator L is $\operatorname{Span}\{1\}$, the subspace of constant polynomials, perhaps as expected (and as we saw in Example 3.1.5 for the differentiation operator on $\mathcal{P}_4$).

EXERCISE

Determine E_0 for the differentiation operator $D\colon \mathcal{P}_n(\mathbb{R}) \to \mathcal{P}_n(\mathbb{R})$.

Thus, we have a computational strategy for finding eigenvectors. What about eigenvalues? Well, they are the numbers λ that satisfy the equation $\det(A - \lambda I) = 0$. Here is another (hopefully familiar) definition.

Definition 3.1.9

Characteristic Polynomial of a Matrix

Let $A \in M_{n\times n}(\mathbb{F})$. The **characteristic polynomial** of A is the polynomial in λ given by $C_A(\lambda) = \det(A - \lambda I)$.

As you may recall, $C_A(\lambda)$ really is a polynomial (see Theorem 3.1.12). Our preceding discussion proves the next proposition.

Proposition 3.1.10

Let $A \in M_{n\times n}(\mathbb{F})$. The eigenvalues of A are the values of λ that are solutions to the equation $\det(A - \lambda I) = 0$. That is, they are the roots of the characteristic polynomial of A.

Example 3.1.11 Going back to Example 3.1.8, with the differentiation operator $D\colon \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ and its standard matrix

$$A = [D]_{\mathcal{S}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

we can compute the characteristic polynomial of A to be

$$C_A(\lambda) = \det(A - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 2 \\ 0 & 0 & -\lambda \end{bmatrix}\right) = -\lambda^3.$$

The only root is $\lambda = 0$, meaning: the only eigenvalue of D is $\lambda = 0$ as we'd already seen.

To close this section, let's recall a few facts about the characteristic polynomial. The important fact here is that this really is a polynomial.

Theorem 3.1.12 If A is an $n \times n$ matrix with entries in $\mathbb{F}$, then the characteristic polynomial of A is a polynomial of degree n with coefficients in $\mathbb{F}$.

This proof is a bit tricky, requiring some finesse with the cofactor expansion definition of the determinant. We shall not write it down here. If you are motivated to try it yourself, first try proving the statement when $n = 2$ and $n = 3$.

Example 3.1.13 In the preceding example, we saw that the characteristic polynomial of the 3×3 matrix $A = [D]_{\mathcal{S}}$ is the degree 3 polynomial $C_A(\lambda) = -\lambda^3$.

Using the fundamental theorem of algebra, i.e., the fact that every degree n polynomial with coefficients in $\mathbb{C}$ has n (possibly repeated) roots in $\mathbb{C}$, we obtain the next result. (For this, note that a polynomial with coefficients in $\mathbb{R}$ also has coefficients in $\mathbb{C}$ since $\mathbb{R} \subseteq \mathbb{C}$.)

Corollary 3.1.14 Let $A \in M_{n\times n}(\mathbb{F})$. Then A has n (possibly repeated) eigenvalues in $\mathbb{C}$.

REMARK

Be careful to remember that we are only guaranteed to find n eigenvalues in $\mathbb{C}$—and not necessarily in $\mathbb{R}$, even if our matrix is in $M_{n\times n}(\mathbb{R})$. For example, $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has no eigenvalues in $\mathbb{R}$, but does indeed have two eigenvalues in $\mathbb{C}$. (What are they?)

Example 3.1.15 If our characteristic polynomial is $C_A(\lambda) = -\lambda^3 = -(\lambda - 0)^3$, then the root 0 is *repeated with multiplicity* 3.

If instead we had $C_A(\lambda) = (\lambda - 1)(\lambda - (1 + i))^2(\lambda + 5)^4$, then we would say that the root 1 is not repeated (or *repeated with multiplicity 1*), while the roots $1 + i$ and -5 are *repeated with multiplicities* 2 *and* 4, respectively.

By thinking carefully about polynomials and how the roots relate to the coefficients, you can prove the following corollary.

Corollary 3.1.16 Let $A \in M_{n\times n}(\mathbb{F})$.

(a) The determinant of A is the product of the complex eigenvalues of A, where each eigenvalue is repeated according to its multiplicity.

(b) The trace of A is the sum of the complex eigenvalues of A, where each eigenvalue is repeated according to its multiplicity.

(We will be able to give a very easy proof of this in Chapter 5. See Example 5.2.3.)

Example 3.1.17 For

$$A = [D]_{\mathcal{S}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

where $C_A(\lambda) = -\lambda^3$, we can indeed verify that

$$\det(A) = 0 \cdot 0 \cdot 0$$

and

$$\operatorname{tr}(A) = 0 + 0 + 0.$$

This example is particularly trivial because the matrix A is upper-triangular, which if you recall means the diagonal entries of A are precisely the eigenvalues of A (repeated according to multiplicity).

It is perhaps more interesting to consider the matrix of D with respect to another basis. For instance, if we take $\mathcal{B} = \{1 + x^2, x, 1 + x - x^2\}$, then we'd find that

$$B = [D]_{\mathcal{B}} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 2 & -\frac{1}{2} & -\frac{5}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

We'll leave it to you to check that $C_B(\lambda) = -\lambda^3$, so that the only eigenvalue of B is 0 repeated with multiplicity 3, and that

$$\det(B) = 0 = 0 \cdot 0 \cdot 0 \quad \text{and} \quad \operatorname{tr}(B) = -\frac{1}{2} + \frac{1}{2} = 0 + 0 + 0.$$

Notice that the matrices A and B have the same characteristic polynomial (and therefore the same determinant and trace). This is not a coincidence! We will explore this in the next section.

EXERCISE

Verify the claims about the matrix B in the preceding example.

3.2 Diagonalization

The fundamental definition in this section is the following.

Definition 3.2.1
Diagonalizable Operator, Diagonalizes

Let V be a finite-dimensional vector space over $\mathbb{F}$. A linear operator $L\colon V \to V$ is **diagonalizabile** if there exists an ordered basis $\mathcal{D}$ for V such that $[L]_{\mathcal{D}}$ is a diagonal matrix. We say that the basis $\mathcal{D}$ **diagonalizes** L.

The point being: an operator L is diagonalizable if there is some coordinate system in which the action of L is easy to interpret.

Example 3.2.2

As noted in the opening to this chapter, the linear operator $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \frac{2(x+y+z)}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

is diagonalizable. Indeed, if we take the ordered basis

$$\mathcal{D} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

for $\mathbb{R}^3$, we find that

$$[L]_{\mathcal{D}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We can use this to interpret L as a reflection in the coordinate system provided by the basis $\mathcal{D}$ for $\mathbb{R}^3$.

How do we determine if a linear operator $L\colon V \to V$ is diagonalizable? And if it is, how do we diagonalize it? That is, how do we determine a basis $\mathcal{D}$ for V such that $[L]_{\mathcal{D}}$ is diagonal?

To answer these questions we will begin by considering $[L]_{\mathcal{B}}$ for an arbitrary basis $\mathcal{B}$. We would like to understand how these matrices, for the various bases of V, are related.

Proposition 3.2.3

Let $L\colon V \to V$ be a linear operator, and let $\mathcal{B}$ and $\mathcal{C}$ be ordered bases for V. Then

$$[L]_{\mathcal{B}} = ({}_{\mathcal{C}}\mathcal{I}_{\mathcal{B}})^{-1}\, [L]_{\mathcal{C}}\, {}_{\mathcal{C}}\mathcal{I}_{\mathcal{B}}.$$

Proof: Since $({}_{\mathcal{C}}\mathcal{I}_{\mathcal{B}})^{-1} = {}_{\mathcal{B}}\mathcal{I}_{\mathcal{C}}$ (by Proposition 2.4.4), and since $[L]_{\mathcal{C}} = {}_{\mathcal{C}}[L]_{\mathcal{C}}$ and $[L]_{\mathcal{B}} = {}_{\mathcal{B}}[L]_{\mathcal{B}}$, this follows from Proposition 2.4.6. □

This proposition motivates our next definition.

Definition 3.2.4
Similar

If B and C are $n \times n$ matrices such that $B = P^{-1}CP$ for some invertible matrix P in $M_{n\times n}(\mathbb{F})$, then we say B **is similar to** C over $\mathbb{F}$.

EXERCISE

Let $A, B, C \in M_{n\times n}(\mathbb{F})$.

(a) Show that A is similar to A.

(b) Show that if A is similar to B then B is similar to A.

(c) Show that if A is similar to B and if B is similar to C then A is similar to C.

EXERCISE

Let $A, B \in M_{n\times n}(\mathbb{F})$. Show that if A is similar to B then A and B have the same characteristic polynomial, eigenvalues, determinant, trace, rank and nullity.

Proposition 3.2.3 says $[L]_{\mathcal{B}}$ is similar to $[L]_{\mathcal{C}}$, with P being the change of basis matrix ${}_{\mathcal{C}}\mathcal{I}_{\mathcal{B}}$.

Example 3.2.5

For $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ as in Example 3.2.2, if we take the standard basis

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

for $\mathbb{R}^3$, then

$$[L]_{\mathcal{S}} = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}.$$

If $\mathcal{D}$ is as in Example 3.2.2 then the change of basis matrix from $\mathcal{D}$ to $\mathcal{S}$ is given by

$${}_{\mathcal{S}}\mathcal{I}_{\mathcal{D}} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

and you can check that

$$({}_{\mathcal{S}}\mathcal{I}_{\mathcal{D}})^{-1}\, [L]_{\mathcal{S}}\; {}_{\mathcal{S}}\mathcal{I}_{\mathcal{D}} = [L]_{\mathcal{D}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So to determine if L is diagonalizable, what we can do is look at the matrix $A = [L]_{\mathcal{B}}$ for any basis $\mathcal{B}$ and then ask if A is similar to a diagonal matrix. We can therefore translate our diagonalizability problem into a problem about matrices.

Definition 3.2.6 **Diagonalizable Matrix, Diagonalizes**

A matrix $A \in M_{n\times n}(\mathbb{F})$ is **diagonalizable (over $\mathbb{F}$)** if there exists an invertible matrix $P \in M_{n\times n}(\mathbb{F})$ such that $P^{-1}AP = D$ where D is a diagonal matrix. We say that the matrix P **diagonalizes** A.

REMARK

You will recall that the field $\mathbb{F}$ plays an important role here. For instance, a matrix A in $M_{n\times n}(\mathbb{R})$ might not be diagonalizable if we insist on having $P \in M_{n\times n}(\mathbb{R})$. However, we might be able to find a suitable P in $M_{n\times n}(\mathbb{C})$. The standard example of this is a 2×2 rotation matrix, which, if the angle of rotation isn't a multiple of π, is not diagonalizable over $\mathbb{R}$ but is diagonalizable over $\mathbb{C}$. (Do you remember why?)

So the question now becomes: When is a matrix A diagonalizable? This is a problem that you have studied in a previous course. We will quickly review—without proof—the solution to this problem.

If we think about how the matrix of a linear map works, then we wish to find a basis of eigenvectors.

Theorem 3.2.7

An $n \times n$ matrix A is diagonalizable if and only if there exists a basis $\mathcal{D} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ for $\mathbb{F}^n$ such that each $\vec{v}_i$ is an eigenvector for A.

If such a basis $\mathcal{D}$ exists, and if we let $P = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$ be the matrix whose columns are the vectors in $\mathcal{D}$, then

$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

where λ_i is the eigenvalue corresponding to the eigenvector $\vec{v}_i$.

So, if we are to diagonalize a matrix, we need to find a basis for $\mathbb{F}^n$ consisting entirely of eigenvectors. Let's take a look at some examples.

Example 3.2.8

Let $L\colon \mathcal{P}_2(\mathbb{F}) \to \mathcal{P}_2(\mathbb{F})$ be the differentiation operator defined by $L(p(x)) = p'(x)$. If $\mathcal{S} = \{1, x, x^2\}$ is the standard basis of $\mathcal{P}_2(\mathbb{F})$ then we have seen in Examples 3.1.8 and 3.1.11 that

$$A = [L]_{\mathcal{S}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

and that $C_A(\lambda) = -\lambda^3$. So the only eigenvalue of L is $\lambda = 0$ and we saw that the corresponding eigenspace is

$$E_0 = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

So there can be no basis of $\mathbb{F}^3$ consisting entirely of eigenvectors of A, since there is *at most* one linearly independent eigenvector of A in $\mathbb{F}^3$. Thus A (and hence L) is **not** diagonalizable.

Example 3.2.9

Let $L\colon \mathcal{P}_1(\mathbb{F}) \to \mathcal{P}_1(\mathbb{F})$ be defined by $L(a+bx) = (a+2b) + (2a+b)x$. Then, using the standard basis $\mathcal{S} = \{1, x\}$, we have that

$$A = [L]_{\mathcal{S}} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

The characteristic polynomial of A is $C_A(\lambda) = (1-\lambda)^2 - 4 = (\lambda - 3)(\lambda + 1)$ and so the eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = -1$.

Using our usual method for finding nullspaces, we obtain

$$E_{\lambda_1} = \text{Null}(A - 3I) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

and

$$E_{\lambda_2} = \text{Null}(A - (-I)) = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}.$$

Since the two eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are linearly independent, we can take $\mathcal{D} = \{\vec{v}_1, \vec{v}_2\}$ as a basis for $\mathbb{F}^2$. If we let $P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ then

$$P^{-1}AP = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}.$$

So we have diagonalized the matrix A. We can convert back from coordinate vectors to polynomials to obtain a basis for $\mathcal{P}_2(\mathbb{F})$. Recall that we are using the standard basis $\mathcal{S} = \{1, x\}$, so $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ correspond to the polynomials $1 + x$ and $1 - x$, respectively. Our work above amounts to the fact that if we let $\mathcal{D} = \{1 + x, 1 - x\}$ then $[L]_{\mathcal{D}} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$.

In the previous example we could check by inspection that the eigenvectors $\vec{v}_1$ and $\vec{v}_2$ corresponding to the two distinct eigenvalues λ_1 and λ_2 are linearly independent. In fact, this is a special case of:

Proposition 3.2.10

Suppose $\lambda_1, \ldots, \lambda_k$ are *distinct* eigenvalues of a square matrix A with corresponding eigenvectors $\vec{v}_1, \ldots, \vec{v}_k$. Then $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly independent.

Combining this with Theorem 3.2.7, we obtain the following useful proposition.

Proposition 3.2.11

If an $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable.

Warning: The converse is false. There are plenty of diagonalizable matrices that do not have distinct eigenvalues. For instance, the $n \times n$ identity matrix when $n > 1$.

Example 3.2.12

Let $L\colon M_{1\times 2}(\mathbb{F}) \to M_{1\times 2}(\mathbb{F})$ be defined by $L(\begin{bmatrix} x & y \end{bmatrix}) = \begin{bmatrix} y & -2x-3y \end{bmatrix}$. Then with respect to the standard basis $\mathcal{S}$ of $M_{1\times 2}(\mathbb{F})$ we have

$$A = [L]_{\mathcal{S}} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}.$$

The characteristic polynomial of A is $C_A(\lambda) = \lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2)$. Thus, A has distinct eigenvalues, and therefore must be diagonalizable.

Going further, we can show that $\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$ and $\left\{\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right\}$ are bases for the eigenspaces E_{-1} and E_{-2}, respectively, and therefore $\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right\}$ is a basis for $\mathbb{F}^2$ consisting entirely of eigenvectors of A So, by Theorem 3.2.7, we must have $P^{-1}AP = D$ where

$$P = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}.$$

Let's check this. We have

$$AP = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & -4 \end{bmatrix}$$

and

$$PD = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & -4 \end{bmatrix}$$

so $P^{-1}AP = D$.

Translating back to $M_{1\times 2}(\mathbb{F})$, we can conclude that the basis $\mathcal{D} = \left\{\begin{bmatrix} -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 2 \end{bmatrix}\right\}$ diagonalizes L.

If the matrix A has repeated eigenvalues then, as you might recall, their multiplicities play an important role in the problem of diagonalizability.

Definition 3.2.13

Algebraic Multiplicity, Geometric Multiplicity

Let $A \in M_{n\times n}(\mathbb{F})$ and let λ be an eigenvalue of A. The **algebraic multiplicity** of λ is the multiplicity of λ as a root of the characteristic polynomial of A. The **geometric multiplicity** of λ is defined to be the dimension of the eigenspace $E_\lambda = \operatorname{Null}(A - \lambda I)$.

The geometric multiplicity of an eigenvalue is really what we are interested in. It tells us how many linearly independent eigenvectors an eigenvalue can have. After all, we want to find a basis consisting entirely of eigenvectors, so we want to be able to find sufficiently many linearly independent eigenvectors. The obvious thing to do is to find bases for each of the eigenspaces and then combine them together (i.e. take their union) to form a set $\mathcal{D}$. Two potential issues arise:

1. Is the resulting set $\mathcal{D}$ linearly independent?

2. Are there enough vectors in $\mathcal{D}$ to span all of $\mathbb{F}^n$?

Amazingly, the answer to question 1 is always yes! (Compare to Proposition 3.2.10.)

Proposition 3.2.14

Suppose $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues of a matrix $A \in M_{n\times n}(\mathbb{F})$, and let $\{\vec{v}_{i,1}, \vec{v}_{i,2}, \ldots, \vec{v}_{i,m_i}\}$ be a basis for the eigenspace corresponding to λ_i (so the dimension of the eigenspace corresponding to λ_i is m_i). Then

$$\{\vec{v}_{1,1}, \vec{v}_{1,2}, \ldots, \vec{v}_{1,m_1}, \vec{v}_{2,1}, \ldots, \vec{v}_{2,m_2}, \ldots, \vec{v}_{k,1}, \ldots, \vec{v}_{k,m_k}\}$$

is a linearly independent subset of $\mathbb{F}^n$.

The other key fact here is that the algebraic multiplicity acts as an upper bound on the geometric multiplicity.

Proposition 3.2.15

Let λ be an eigenvalue of an $n \times n$ matrix A. Then

$$1 \leq \text{geometric multiplicity of } \lambda \leq \text{algebraic multiplicity of } \lambda \leq n.$$

Now, since the sum of the algebraic multiplicities is equal to $\deg C_A(\lambda) = n = \dim \mathbb{F}^n$, if the geometric multiplicity is ever strictly less than the algebraic multiplicity for any eigenvalue, we immediately know that we cannot find enough linearly independent eigenvectors to diagonalize the matrix. Conversely, if the geometric multiplicity is equal to the algebraic multiplicity for each eigenvalue, then we will be able to find enough linearly independent eigenvectors.

Theorem 3.2.16

(Characterization of Diagonalizability)

A matrix $A \in M_{n\times n}(\mathbb{F})$ is diagonalizable if and only if every eigenvalue of A has its geometric multiplicity equal to its algebraic multiplicity.

EXERCISE

Supply proofs for as many of the unproved propositions and theorems above as you can. For those you don't know how to prove, look up their proofs in the notes from your previous linear algebra course.

This theorem completely answers the question of whether or not an $n \times n$ matrix is diagonalizable. Let us summarize all our previous results.

ALGORITHM (Diagonalization of an Operator)

To diagonalize a linear operator $L: V \to V$:

1. Pick any basis $\mathcal{B}$ for V and determine the matrix $A = [L]_{\mathcal{B}}$.

2. Compute and factor the characteristic polynomial $C_A(\lambda)$ to find the eigenvalues $\lambda_1, \ldots, \lambda_k$ of A. Let a_i denote the algebraic multiplicity of λ_i.

3. Determine a basis $\mathcal{B}_i$ for the eigenspace E_{λ_i}, for each $i = 1, \ldots, k$. Let $g_i = \dim E_{\lambda_i}$ denote the geometric multiplicity of λ_i.

4. A (hence L) is diagonalizable if and only if $a_i = g_i$ for all $i = 1, \ldots, k$.

5. If A is diagonalizable, then $\mathcal{D} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k$ is a basis for $\mathbb{F}^n$ (where $n = \dim V$) consisting of eigenvectors of A. If P is the matrix whose columns are the vectors in $\mathcal{D}$, then $D = P^{-1}AP$ is a diagonal matrix. The diagonal entries of D are λ_1 (listed a_1 times), $\ldots, \lambda_k$ (listed a_k times). The order of eigenvalues matches the order in which their corresponding eigenvectors occur as columns in P.

6. To determine a basis for V that diagonalizes L, take each of the vectors in $\mathcal{D}$, view it as a coordinate vector in $\mathbb{F}^n$ with respect to the basis $\mathcal{B}$ from Step 1, and thereby convert it into vector V. The set of all these vectors is then the desired basis for V.

Example 3.2.17 Let $L\colon M_{2\times 2}(\mathbb{F}) \to M_{2\times 2}(\mathbb{F})$ be defined by

$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 5a+2b+d & -2a+b-d \\ 4a+4b+3c+2d & 16a-8c-5d \end{bmatrix}.$$

Using the standard basis $\mathcal{S}$ of $M_{2\times 2}(\mathbb{F})$, we have

$$A = [L]_{\mathcal{S}} = \begin{bmatrix} 5 & 2 & 0 & 1 \\ -2 & 1 & 0 & -1 \\ 4 & 4 & 3 & 2 \\ 16 & 0 & -8 & -5 \end{bmatrix}.$$

Then $C_A(\lambda) = -(\lambda-3)^3(\lambda+5)$. So the eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = -5$ with algebraic multiplicities $a_1 = 3$ and $a_2 = 1$, respectively.

We must now determine the geometric multiplicities. The geometric multiplicity g_2 is easy: since $1 \le g_2 \le a_2 = 1$, we must have that $g_2 = 1$.

For g_1, we need to find the dimension of the eigenspace corresponding to $\lambda_1 = 3$. We have

$$E_{\lambda_1} = \operatorname{Null}(A - 3I) = \operatorname{Null}\left(\begin{bmatrix} 2 & 2 & 0 & 1 \\ -2 & -2 & 0 & -1 \\ 4 & 4 & 0 & 2 \\ 16 & 0 & -8 & -8 \end{bmatrix}\right).$$

Row reduction leads to

$$\begin{bmatrix} 2 & 2 & 0 & 1 \\ -2 & -2 & 0 & -1 \\ 4 & 4 & 0 & 2 \\ 16 & 0 & -8 & -8 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 0 & 1 \\ 0 & 16 & 8 & 16 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The nullity of these matrices is 2. Hence, $g_1 = \dim E_{\lambda_1} = \operatorname{nullity}(A - 3I) = 2$. However, $a_1 = 3$. So, since $g_1 \ne a_1$, we conclude that A (and hence L) is not diagonalizable.

Example 3.2.18 Let $L\colon \mathcal{P}_3(\mathbb{F}) \to \mathcal{P}_3(\mathbb{F})$ be defined by

$$L(a+bx+cx^2+dx^3) = (-2b+2c) + (-2a+2d)x + (2a-2d)x^2 + (2b-2c)x^3.$$

Using the standard basis $\mathcal{S}$ of $\mathcal{P}_3(\mathbb{F})$, we have

$$A = [L]_{\mathcal{S}} = \begin{bmatrix} 0 & -2 & 2 & 0 \\ -2 & 0 & 0 & 2 \\ 2 & 0 & 0 & -2 \\ 0 & 2 & -2 & 0 \end{bmatrix}.$$

Then $C_A(\lambda) = \lambda^2(\lambda-4)(\lambda+4)$. So the eigenvalues of A are $\lambda_1 = 0$, $\lambda_2 = 4$ and $\lambda_3 = -4$, with algebraic multiplicities $a_1 = 2$, $a_2 = a_3 = 1$. Just as in the previous example, we can immediately conclude that $g_2 = g_3 = 1$.

So it remains to determine $g_1 = \dim E_{\lambda_1} = \text{nullity}(A - 0I) = \text{nullity}(A)$. A quick row reduction of A leads to

$$\begin{bmatrix} 0 & -2 & 2 & 0 \\ -2 & 0 & 0 & 2 \\ 2 & 0 & 0 & -2 \\ 0 & 2 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, $\text{nullity}(A) = 2$. So $g_1 = a_1 = 2$ and therefore A (hence L) is diagonalizable.

From this alone we know that A must be similar to the diagonal matrix $D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$.

But let's actually find a diagonalizing basis. For this, we need to find bases for E_0, E_{-4} and E_4.

Going through our usual row reduction process (steps omitted), we find that

$$E_0 = \text{Null}(A - 0I) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\},$$

$$E_4 = \text{Null}(A - 4I) = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right\},$$

and

$$E_{-4} = \text{Null}(A + 4I) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\}.$$

Therefore,

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\}$$

is a basis for $\mathbb{F}^4$ that diagonalizes A. If we view these vectors as being coordinate vectors with respect to the standard basis $\mathcal{S} = \{1, x, x^2, x^3\}$, we can convert them to obtain the basis $\mathcal{D} = \{1+x^3, x+x^2, 1-x+x^2-x^3, 1+x-x^2-x^3\}$ that diagonalizes L. Let's check directly that

$$[L]_{\mathcal{D}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

We have

$$L(1+x^3) = 0, \quad \text{so } [L(1+x^3)]_{\mathcal{D}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$L(x+x^2) = 0, \quad \text{so } [L(x+x^2)]_{\mathcal{D}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$L(1-x+x^2-x^3) = 4(1-x+x^2-x^3), \quad \text{so } [L(1-x+x^2-x^3)]_{\mathcal{D}} = \begin{bmatrix} 0 \\ 0 \\ 4 \\ 0 \end{bmatrix}$$

$$L(1+x-x^2-x^3) = -4(1+x-x^2-x^3), \quad \text{so } [L(1+x-x^2-x^3)]_{\mathcal{D}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -4 \end{bmatrix}.$$

Example 3.2.19

Let $L\colon \mathcal{P}_1(\mathbb{F}) \to \mathcal{P}_1(\mathbb{F})$ be defined by

$$L(a+bx) = (a+2b) + (b-2a)x.$$

Using the standard basis $\mathcal{S}$ of $\mathcal{P}_1(\mathbb{F})$, we have

$$A = [L]_{\mathcal{S}} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}.$$

Then $C_A(\lambda) = \lambda^2 - 2\lambda + 5$. If $\mathbb{F} = \mathbb{R}$ then there are no real eigenvalues, and so no hope for diagonalization!

If $\mathbb{F} = \mathbb{C}$ then $C_A(\lambda)$ factors into $(\lambda - (-1+2i))(\lambda - (-1-2i))$ and so A has two distinct complex eigenvalues $\lambda_1 = -1+2i$ and $\lambda_2 = -1-2i$. Thus, A is diagonalizable. We leave it to you to find a diagonalizing basis for L.

EXERCISE

Complete the previous example and find a basis $\mathcal{D}$ for $\mathcal{P}_1(\mathbb{C})$ such that $[L]_{\mathcal{D}}$ is diagonal.

3.3 Applications of Diagonalization

We now know how to figure out whether or not a linear operator $L\colon V \to V$ is diagonalizable, and even better, how to find a basis that diagonalizes L. We have learned that this is equivalent to the problem of determining whether or not a square matrix is diagonalizable.

There are several practical applications of diagonalization. You might be already familiar with one: taking powers of matrices. If A is diagonalizable, say with $A = PDP^{-1}$, then we can quickly compute A^k as

$$\begin{aligned} A^k &= (PDP^{-1})(PDP^{-1})\cdots(PDP^{-1}) \\ &= PD(P^{-1}P)D(P^{-1}P)\cdots(P^{-1}P)DP^{-1} \\ &= PD^kP^{-1}. \end{aligned}$$

This is useful because if

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

then

$$D^k = \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^n & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n^k \end{bmatrix}.$$

So, for instance, to compute A^{1000} we don't need to multiply A with itself 1000 times. We can simply perform *two* multiplications to compute $PD^{1000}P^{-1}$ instead.

Example 3.3.1 Suppose $A = \begin{bmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{bmatrix}$. Then $A = PDP^{-1}$ where

$$P = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Therefore

$$\begin{aligned} A^{100} &= PD^{100}P^{-1} \\ &= \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{100} & 0 \\ 0 & 0 & 2^{100} \end{bmatrix} \begin{bmatrix} 3 & -3 & 1 \\ -3 & 4 & -1 \\ -1 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 3 - 2\cdot 2^{100} & -3 + 3\cdot 2^{100} & 1 - 2^{100} \\ 3 - 3\cdot 2^{100} & -3 + 4\cdot 2^{100} & 1 - 2^{100} \\ 3 + 3\cdot 2^{100} & -3 - 3\cdot 2^{100} & 1 \end{bmatrix} \end{aligned}$$

Taking powers of matrices like this arises when studying dynamical systems or Markov chains in probability theory, to name just a couple. In particular, the Google PageRank algorithm involves these ideas. If you're interested to learn how, check out the video *How does Google Google?* on mathtube.org by Margot Gerritsen.

To close this section, we'll take a look at another application of diagonalization. We'll see some other applications later in this course.

Decoupling Differential Equations

It often comes up when modelling nature that you have a few differential equations where the variables from each equation appear in every other one. These can be extremely difficult to solve, but occasionally we can separate out the variables in a process called *decoupling*. It turns out that if your differential equations take on a very precise form, we can diagonalize a matrix and as a result decouple the equations. Let's see this in an example.

Example 3.3.2 Xavier and Yvonne are in a zombie apocalypse, and they are continuously killing zombies, and the following things are true about the way they kill zombies.

- Both get better with practice. (A reasonable assumption.)
- Both slow down as the other kills zombies. (They stop to congratulate the other person, and to give them a high-five if they are within slapping distance.)
- For every zombie Xavier kills, his kill rate goes up by a factor of 5 and Yvonne's goes down by a factor of 6.
- For every zombie Yvonne kills, her kill rate increases by a factor of 2 and Xavier's decreases by a factor of 6.

So, if we let x be the number of zombies killed by Xavier, y the number killed by Yvonne and t the time since the apocalypse started, we can set up the following system of differential equations that models the situation at hand.

$$\frac{dx}{dt} = 5x - 3y$$
$$\frac{dy}{dt} = -6x + 2y.$$

Seemingly out of nowhere, let's make the substitutions

$$u = -\frac{2}{3}x + \frac{1}{3}y \quad \text{and} \quad w = \frac{1}{3}x + \frac{1}{3}y.$$

We now have

$$\begin{aligned}\frac{du}{dt} &= -\frac{2}{3}\frac{dx}{dt} + \frac{1}{3}\frac{dy}{dt} \\ &= -\frac{2}{3}(5x - 3y) + \frac{1}{3}(-6x + 2y) \\ &= -\frac{16}{3}x + \frac{8}{3}y \\ &= 8u\end{aligned}$$

and

$$\begin{aligned}\frac{dw}{dt} &= \frac{1}{3}(5x-3y)+\frac{1}{3}(-6x+2y)\\ &= -\frac{1}{3}x-\frac{1}{3}y\\ &= -w.\end{aligned}$$

These differential equations are much easier to deal with, and can be easily solved, and then converted back to our x and y variables. That's not the important thing here. The question you should have burning in your mind is, *"how did we choose u and w?"*

To answer this, we do what we do best: bring matrices into the picture!

Let

$$\vec{x}=\begin{bmatrix}x\\ y\end{bmatrix},\quad \frac{d\vec{x}}{dt}=\begin{bmatrix}\frac{dx}{dt}\\ \frac{dy}{dt}\end{bmatrix},\quad \text{and}\quad A=\begin{bmatrix}5 & -3\\ -6 & 2\end{bmatrix}.$$

Then

$$\frac{d\vec{x}}{dt}=A\vec{x}.$$

Now, it turns out that A is diagonalizable. In fact, $P^{-1}AP=D$ where

$$P=\begin{bmatrix}-1 & 1\\ 1 & 2\end{bmatrix}\quad \text{and}\quad D=\begin{bmatrix}8 & 0\\ 0 & -1\end{bmatrix}$$

so

$$\frac{d\vec{x}}{dt}=PDP^{-1}\vec{x}.$$

If we let

$$\vec{u}=\begin{bmatrix}u\\ w\end{bmatrix}=P^{-1}\vec{x}=\begin{bmatrix}-\frac{2}{3}x+\frac{1}{3}y\\ \frac{1}{3}x+\frac{1}{3}y\end{bmatrix}$$

we have $\vec{x}=P\vec{u}$. Putting this back into the differential equation gives

$$\frac{d(P\vec{u})}{dt}=P\frac{d\vec{u}}{dt}=PD\vec{u}.$$

Multiplying on the left by P^{-1} gives

$$\frac{d\vec{u}}{dt}=D\vec{u}$$

or

$$\begin{bmatrix}\frac{du}{dt}\\ \frac{dw}{dt}\end{bmatrix}=\begin{bmatrix}8 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}u\\ w\end{bmatrix}=\begin{bmatrix}8u\\ -w\end{bmatrix}$$

giving us our decoupled differential equations.

The take home message here is that the matrix P told us what substitution to make. This shouldn't be too surprising because P can always be thought of as a change of coordinate matrix—one that changes coordinates from the ones we started with to a more natural set of coordinates depending on the problem at hand.

In general, suppose you have a system of differential equations of the form

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy.$$

Here the variables x and y depend on each other, but it would be great if they didn't, since then you could solve two separate differential equations. If the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is diagonalizable, then by a simple change of coordinates, we can decouple the differential equation.

Let $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\frac{d\vec{x}}{dt} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix}$. Then the set of equations above can be written as $\frac{d\vec{x}}{dt} = A\vec{x}$.

Suppose $P^{-1}AP = D$ for some diagonal matrix $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$. Let $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and let $\vec{u} = \begin{bmatrix} u \\ w \end{bmatrix} = P^{-1}\vec{x}$. Then

$$\frac{d\vec{x}}{dt} = A\vec{x} \quad \Longleftrightarrow \quad P\frac{d\vec{u}}{dt} = PDP^{-1}\vec{x} \quad \Longleftrightarrow \quad \frac{d\vec{u}}{dt} = D\vec{u}.$$

Rewriting this last equation as a pair of differential equations we get

$$\frac{du}{dt} = \lambda_1 u$$
$$\frac{dw}{dt} = \lambda_2 w.$$

Now we have two decoupled equations, each which can be solved independently of the other. This process of decoupling is easily generalized to n variables and n equations.

Chapter 4

Inner Product Spaces

4.1 Inner Products

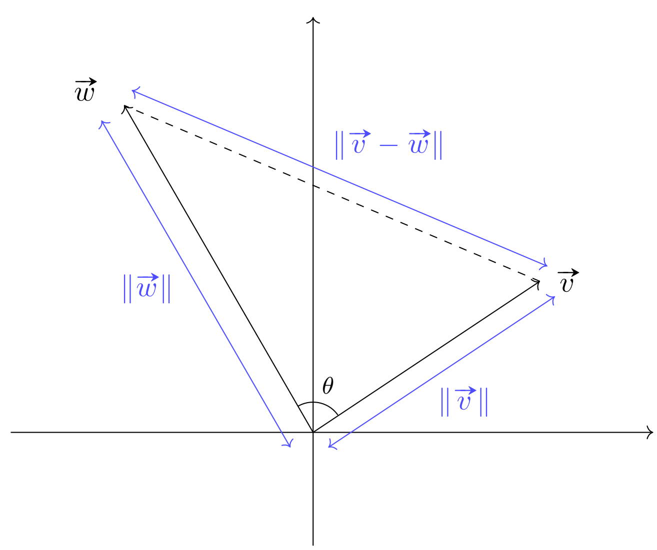
We have seen that if you have a finite-dimensional real vector space, once you pick a basis, you may as well think of the vector space as $\mathbb{R}^n$ (and indeed, any n-dimensional real vector space is isomorphic to $\mathbb{R}^n$). This provides us with with plenty of geometric intuition. A very useful feature of $\mathbb{R}^2$ and $\mathbb{R}^3$ is that they come with well-defined notions of length and angle.

In $\mathbb{R}^2$, we know that the length of a vector $\vec{v} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}^T$ is given by

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2}.$$

This formula is given to us by the Pythagorean theorem.

To compute the angle θ between two non-zero vectors $\vec{v} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}^T$ and $\vec{w} = \begin{bmatrix} w_1 & w_2 \end{bmatrix}^T$, we invoke the cosine rule:



$$\begin{aligned}\cos\theta &= \frac{\|\vec{v}\|^2 + \|\vec{w}\|^2 - \|\vec{v} - \vec{w}\|^2}{2\,\|\vec{v}\|\,\|\vec{w}\|} \\ &= \frac{v_1^2 + v_2^2 + w_1^2 + w_2^2 - (v_1 - w_1)^2 - (v_2 - w_2)^2}{2\,\|\vec{v}\|\,\|\vec{w}\|} \\ &= \frac{2v_1w_1 + 2v_2w_2}{2\,\|\vec{v}\|\,\|\vec{w}\|} \\ &= \frac{v_1w_1 + v_2w_2}{\|\vec{v}\|\,\|\vec{w}\|}.\end{aligned}$$

In $\mathbb{R}^3$, a similar thing occurs. Let $\vec{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^T$ and $\vec{w} = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix}^T$ be two vectors in $\mathbb{R}^3$. Then the length of $\vec{v}$ is

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

and the angle θ between $\vec{v}$ and $\vec{w}$ (if they are non-zero) is given by

$$\cos\theta = \frac{v_1w_1 + v_2w_2 + v_3w_3}{\|\vec{v}\|\,\|\vec{w}\|}.$$

Just like electricity and magnetism are actually two sides of the same coin, angles and lengths in $\mathbb{R}^2$ and $\mathbb{R}^3$ are just two sides of the same coin, and that coin is the dot product.

Recall that the dot product on $\mathbb{R}^n$ is defined as

$$\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = v_1w_1 + \cdots + v_nw_n.$$

So it is a function that takes as input two vectors in $\mathbb{R}^n$ and produces a real number as output. Our expressions above for length and angle can be reformulated as

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} \quad \text{and} \quad \cos\theta = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\,\|\vec{w}\|}$$

where θ is the angle between $\vec{v}$ and $\vec{w}$. Since the dot product is defined on $\mathbb{R}^n$ (and not just $\mathbb{R}^2$ and $\mathbb{R}^3$), we can use this to *define* length and angle in $\mathbb{R}^n$.

If we are to generalize the dot product to other vector spaces, we would want it to satisfy certain properties. Whatever our generalization is, it should be a function

$$\langle\ ,\ \rangle : V \times V \to \mathbb{F}$$

that takes as input two vectors $\vec{v}, \vec{w} \in V$ and produces a real number $\langle \vec{v}, \vec{w} \rangle$ as output. We would then like to use this to define the length of a vector $\vec{v}$ as $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$.

This expression should behave the same way the length does in $\mathbb{R}^2$ and $\mathbb{R}^3$. For example, we would like the length of a vector to be a positive real number, that is $\|\vec{v}\| \geq 0$. We would also want the length of a vector to be zero if and only if the vector itself is the zero vector.

We would then like to define the angle θ between two non-zero vectors $\vec{v}, \vec{w} \in V$ via

$$\cos\theta = \frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{v}\|\,\|\vec{w}\|}$$

just as we did above. For this to make sense, we need to know that the number on the right-side lies in the interval $[-1, 1]$. That is, we would like to have

$$\frac{|\langle \vec{v}, \vec{w} \rangle|}{\|\vec{v}\| \, \|\vec{w}\|} \leq 1.$$

Furthermore, in $\mathbb{R}^2$ and $\mathbb{R}^3$ we know the shortest path between two points is a straight line, which is reflected in the triangle inequality. That is, we would like it to be true if

$$\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\| .$$

Finally, the length of a vector ought to behave nicely under scalar multiplication. That is, would like to have $\|\alpha \vec{v}\| = |\alpha| \, \|\vec{v}\|$.

With these desires in mind, let's define an inner product on an arbitrary vector space. Our definition will produce a notion of length and angle that will meet *all* of our requirements above!

Definition 4.1.1

Inner Product, Conjugate Symmetry, Linearity in First Argument, Positive-Definite

Let V be a vector space over $\mathbb{F}$. An **inner product** on V is a function

$$\langle \ , \ \rangle : V \times V \to \mathbb{F}$$

such that for all $\vec{u}, \vec{v}, \vec{w} \in V$ and $\alpha \in \mathbb{F}$,

1. $\langle \vec{v}, \vec{w} \rangle = \overline{\langle \vec{w}, \vec{v} \rangle}$.
2. $\langle \alpha \vec{v}, \vec{w} \rangle = \alpha \langle \vec{v}, \vec{w} \rangle$.
3. $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$.
4. (a) $\langle \vec{v}, \vec{v} \rangle \geq 0$.
   (b) If $\langle \vec{v}, \vec{v} \rangle = 0$ then $\vec{v} = \vec{0}$.

A vector space V equipped with an inner product $\langle \ , \ \rangle$ is called an **inner product space**.

Property 1 is called **conjugate symmetry**. Properties 2 and 3 together are called **linearity in the first argument**. Property 4 is called **positive definiteness**, and we say that $\langle \ , \ \rangle$ is **positive definite**.

Note that if $\mathbb{F} = \mathbb{R}$, then $\overline{a} = a$ for all $a \in \mathbb{R}$ so the first property becomes $\langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle$.

Let's see some examples.

Example 4.1.2

Of course, the dot product on $\mathbb{R}^n$ satisfies properties 1–4, and so it defines an inner product on $\mathbb{R}^n$. The dot product on $\mathbb{C}^n$, however, is **not** an inner product.

EXERCISE

(a) Carefully check that the dot product is indeed an inner product on $\mathbb{R}^n$.

(b) Show that the dot product on $\mathbb{C}^2$ is not an inner product by finding a vector $\vec{z} \in \mathbb{C}^2$ such that $\vec{z} \cdot \vec{z}$ is a negative real number.

Example 4.1.3 **(Standard inner product on $\mathbb{C}^n$)**

Let $\vec{v}, \vec{w} \in \mathbb{C}^n$ where

$$\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \quad \text{and} \quad \vec{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}.$$

We define the standard inner product on $\mathbb{C}^n$ by

$$\langle \vec{v}, \vec{w} \rangle = v_1\overline{w}_1 + \cdots + v_n\overline{w}_n.$$

Let's prove that the standard inner product is indeed an inner product.

Proof: Let $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, \vec{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}, \vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \in V$ and $\alpha \in \mathbb{C}$. Then

$$\begin{aligned} \langle \vec{v}, \vec{w} \rangle &= v_1\overline{w}_1 + \cdots + v_n\overline{w}_n \\ &= \overline{w}_1 v_1 + \cdots + \overline{w}_n v_n \\ &= \overline{w_1\overline{v}_1 + \cdots + w_n\overline{v}_n} \\ &= \overline{\langle \vec{w}, \vec{v} \rangle} \end{aligned}$$

so property 1 holds. We have

$$\begin{aligned} \langle \alpha\vec{v}, \vec{w} \rangle &= \alpha v_1\overline{w}_1 + \cdots + \alpha v_n\overline{w}_n \\ &= \alpha(v_1\overline{w}_1 + \cdots + v_n\overline{w}_n) \\ &= \alpha \langle \vec{v}, \vec{w} \rangle \end{aligned}$$

and

$$\begin{aligned} \langle \vec{u} + \vec{v}, \vec{w} \rangle &= (u_1 + v_1)\overline{w}_1 + \cdots + (u_n + v_n)\overline{w}_n \\ &= u_1\overline{w}_1 + v_1\overline{w}_1 + \cdots + u_n\overline{w}_n + v_n\overline{w}_n \\ &= (u_1\overline{w}_1 + \cdots + u_n\overline{w}_n) + (v_1\overline{w}_1 + \cdots + v_n\overline{w}_n) \\ &= \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle \end{aligned}$$

so properties 2 and 3 hold. For 4(a) we have

$$\langle \vec{v}, \vec{v} \rangle = v_1\overline{v}_1 + \cdots + v_n\overline{v}_n = |v_1|^2 + \cdots + |v_n|^2 \geq 0.$$

Finally, suppose $\langle \vec{v}, \vec{v} \rangle = |v_1|^2 + \cdots + |v_n|^2 = 0$. Since each $|v_i|^2$ is a postive real number, the only way this can be true is if $|v_1| = \cdots = |v_n| = 0$. This in turn implies $v_1 = \cdots = v_n = 0$ so $\vec{v} = \vec{0}$.

Thus, the standard inner product on $\mathbb{C}^n$ is indeed an inner product. $\square$

Example 4.1.4 **(Standard inner product on $\mathbb{F}^n$)**

For $\vec{v}, \vec{w} \in \mathbb{F}^n$, where

$$\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \quad \text{and} \quad \vec{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix},$$

we define the standard inner by

$$\langle \vec{v}, \vec{w} \rangle = v_1 \overline{w}_1 + \cdots + v_n \overline{w}_n.$$

If $\mathbb{F} = \mathbb{R}$, this gives the dot product, and if $\mathbb{F} = \mathbb{C}$ this gives the standard inner product on $\mathbb{C}^n$ defined in the previous example.

Example 4.1.5 For $p, q \in \mathcal{P}_n(\mathbb{R})$, define

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)\, dx.$$

This is an inner product on $\mathcal{P}_n(\mathbb{R})$. We will check property 4 and leave the rest as an exercise.

If $p \in \mathcal{P}_n(\mathbb{R})$ then

$$\langle p, p \rangle = \int_{-1}^{1} (p(x))^2\, dx$$

is non-negative because it is the integral of a non-negative function. Moreover, as we know from calculus, the integral of a non-negative continuous function (such as a polynomial) over an interval is zero if and only if that function is itself zero on the interval. This shows that $\langle p, p \rangle = 0$ if and only if p is the zero polynomial.

EXERCISE

Check that $\langle\ ,\ \rangle$ defined in the previous example satisfies properties 1–3 and therefore is an inner product on $\mathcal{P}_3(\mathbb{R})$.

Example 4.1.6 **(Standard inner product on $M_{m\times n}(\mathbb{R})$)**

For $A, B \in M_{m\times n}(\mathbb{R})$, define

$$\langle A, B \rangle = \operatorname{tr}(B^T A).$$

This is an inner product on $M_{m\times n}(\mathbb{R})$. If $A = [a_{ij}]$ and $B = [b_{ij}]$, then the above expression can be expanded as

$$\langle A, B \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ij}.$$

EXERCISE

(a) Verify that $\langle\ ,\ \rangle$ defined in the preceding example is an inner product on $M_{m\times n}(\mathbb{R})$.

(b) What does this inner product give if $n=1$ (so that $M_{m\times n}(\mathbb{R})=\mathbb{R}^m$)?

EXERCISE (Standard inner product on $M_{m\times n}(\mathbb{C})$)

For a matrix $A\in M_{m\times n}(\mathbb{C})$, we define its **adjoint** (or **conjugate-transpose**) to be the matrix $A^*=\overline{A^T}$ in $M_{n\times m}(\mathbb{C})$. We will return to this notion in Chapter 5.

(a) Show that $\langle A,B\rangle=\operatorname{tr}(B^*A)$ defines an inner product on $M_{m\times n}(\mathbb{C})$.

(b) Show that $\langle A,B\rangle=\operatorname{tr}(A^*B)$ does **not** define an inner product on $M_{m\times n}(\mathbb{C})$.

Since the inner product of a vector with itself is always a positive real number, it now makes sense to define the length of a vector.

Definition 4.1.7
Norm, Length

Let $\vec{v}$ be a vector in an inner product space V. The **norm** (or **length**) of $\vec{v}$ is defined by

$$\|\vec{v}\|=\sqrt{\langle\vec{v},\vec{v}\rangle}.$$

This definition of "length" satisfies many of the properties one would expect length to satisfy. See the following exercise and Proposition 4.2.7.

EXERCISE

Prove that, in any inner product space, $\|\vec{0}\|=0$.

Example 4.1.8

On $\mathcal{P}_2(\mathbb{C})$ consider

$$\langle p,q\rangle=p(i)\overline{q(i)}+p(-i)\overline{q(-i)}+p(1)\overline{q(1)}.$$

This is an inner product. Indeed, it is left as an exercise to check properties 1, 2, and 3. For 4a,

$$\langle p,p\rangle=|p(i)|^2+|p(-i)|^2+|p(1)|^2\geq 0.$$

For 4b, suppose $\langle p,p\rangle=0$. Then

$$|p(i)|^2+|p(-i)|^2+|p(1)|^2=0$$

so $p(i)=p(-i)=p(1)=0$. Since p is a polynomial of degree at most 2 that evaluates to 0 at three distinct points, we must have $p=0$. Alternatively, suppose $p=ax^2+bx+c$.

Then $p(i) = p(-i) = p(1) = 0$ gives the system of equations

$$\begin{aligned} -a + bi + c &= 0 \\ -a - bi + c &= 0 \\ a + b + c &= 0. \end{aligned}$$

Plugging this into an augmented matrix and row-reducing, we get

$$\left[\begin{array}{ccc|c} -1 & i & 1 & 0 \\ -1 & -i & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

so $a = b = c = 0$ and therefore $p = 0$.

So this is an inner product. With respect to this inner product, let's compute the norm of some vectors. We have

$$\|1\| = \sqrt{\langle 1, 1\rangle} = \sqrt{3}, \quad \|x\| = \sqrt{\langle x, x\rangle} = \sqrt{|i|^2 + |-i|^2 + |1|^2} = \sqrt{3}$$

and

$$\|1 + x\| = \sqrt{\langle 1 + x, 1 + x\rangle} = \sqrt{|1 + i|^2 + |1 - i|^2 + |1 + 1|^2} = \sqrt{8}.$$

Notice that $\|1 + x\| \neq \|1\| + \|x\|$.

It may be natural to ask at this point whether or not every vector space can be turned into an inner product space. Let's answer that now.

Proposition 4.1.9 Every finite-dimensional vector space admits an inner product.

Proof: Let V be a vector space with basis $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$. Let $\vec{v} = t_1\vec{v}_1 + \cdots + t_n\vec{v}_n$ and $\vec{w} = s_1\vec{v}_1 + \cdots + s_n\vec{v}_n$. Then it is left as an exercise to check

$$\langle \vec{v}, \vec{w}\rangle = t_1\overline{s_1} + \cdots + t_n\overline{s_n}$$

is an inner product on V. □

EXERCISE

Complete the proof of Proposition 4.1.9.

It is true that every infinite-dimensional vector space also admits an inner product, but we won't be proving that here.

To finish off this introduction to inner products, let's state a few of their important properties.

Proposition 4.1.10 Let V be an inner product space. For all $\vec{v}, \vec{u}, \vec{w} \in V$ and $\alpha \in \mathbb{F}$ the following properties are true.

(a) $\langle \vec{0}, \vec{v} \rangle = \langle \vec{v}, \vec{0} \rangle = 0$.

(b) $\langle \vec{v}, \alpha\vec{w} \rangle = \overline{\alpha} \langle \vec{v}, \vec{w} \rangle$.

(c) $\langle \vec{v}, \vec{u} + \vec{w} \rangle = \langle \vec{v}, \vec{u} \rangle + \langle \vec{v}, \vec{w} \rangle$.

Proof: We will prove part (a). There are a couple of ways to do this. Perhaps the slickest is to note that $\vec{0} = 0 \cdot \vec{0}$. So linearity in the first argument gives

$$\langle \vec{0}, \vec{v} \rangle = \langle 0 \cdot \vec{0}, \vec{v} \rangle = 0 \langle \vec{0}, \vec{v} \rangle = 0.$$

This proves half of part (a). For the second half, we use conjugate-symmetry:

$$\langle \vec{v}, \vec{0} \rangle = \overline{\langle \vec{0}, \vec{v} \rangle} = \overline{0} = 0.$$

Parts (b) and (c) can be proved quickly by appealing to conjugate symmetry. We'll leave the details as an exercise. □

EXERCISE

Prove parts (b) and (c) of Proposition 4.1.10.

4.2 Orthogonality

Recall in $\mathbb{R}^n$ that the angle θ between two non-zero vectors $\vec{v}$ and $\vec{w}$ is given by

$$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \, \|\vec{w}\|}.$$

Therefore if the dot product of two vectors is 0, we know they are perpendicular. The notion of being perpendicular (or *orthogonal* as we will call it) turns out to be an extremely useful notion in inner product spaces.

Definition 4.2.1
Orthogonal, $\perp$

Let V be an inner product space. We say $\vec{v}$ is **orthogonal** to $\vec{w}$, and write $\vec{v} \perp \vec{w}$, if $\langle \vec{v}, \vec{w} \rangle = 0$.

Notice that $\langle \vec{v}, \vec{w} \rangle = 0$ if and only if $\langle \vec{w}, \vec{v} \rangle = 0$ (why?) so the definition is symmetric in $\vec{v}$ and $\vec{w}$. That is, we're safe to say that $\vec{v}$ and $\vec{w}$ are orthogonal, instead of $\vec{v}$ is orthogonal to $\vec{w}$ or vice versa.

Example 4.2.2

Consider $\mathcal{P}_2(\mathbb{R})$ with the inner product

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x) \, dx.$$

Then

$$\langle 1, x \rangle = \int_{-1}^{1} x \, dx = 0$$

so 1 and x are orthogonal. However,

$$\langle 1, x^2 \rangle = \int_{-1}^{1} x^2 \, dx = \frac{2}{3}$$

so 1 and x^2 are not orthogonal.

One thing we know from geometry is that if we have a right-angled triangle, then Pythagoras' theorem holds. So, if we are to believe that being orthogonal really means that two vectors are at right angles to each other, we should expect the Pythagorean theorem to hold. Indeed it does!

Proposition 4.2.3 **(Pythagorean Theorem)**

Let V be an inner product space. Suppose $\vec{v} \perp \vec{w}$. Then $\|\vec{v} + \vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2$.

Proof: We have

$$\begin{aligned} \|\vec{v} + \vec{w}\|^2 &= \langle \vec{v} + \vec{w}, \vec{v} + \vec{w} \rangle \\ &= \langle \vec{v}, \vec{v} + \vec{w} \rangle + \langle \vec{w}, \vec{v} + \vec{w} \rangle \\ &= \langle \vec{v}, \vec{v} \rangle + \langle \vec{v}, \vec{w} \rangle + \langle \vec{w}, \vec{v} \rangle + \langle \vec{w}, \vec{w} \rangle \\ &= \langle \vec{v}, \vec{v} \rangle + \langle \vec{w}, \vec{w} \rangle \\ &= \|\vec{v}\|^2 + \|\vec{w}\|^2, \end{aligned}$$

completing the proof. □

Our short term goal is to prove the Cauchy–Schwarz inequality (Theorem 4.2.5 below), which will allow us to define the angle between two vectors in an arbitrary inner product space. To do that we first need the following technical lemma.

Lemma 4.2.4

Let V be an inner product space and let $\vec{v}, \vec{w} \in V$ such that $\vec{w} \neq \vec{0}$. Then $\vec{w}$ is orthogonal to $\vec{v} - \frac{\langle \vec{v}, \vec{w} \rangle}{\langle \vec{w}, \vec{w} \rangle}\vec{w}$.

Proof: We simply need to take the inner product between these two vectors and show it is zero. We have

$$\begin{aligned} \left\langle \vec{v} - \frac{\langle \vec{v}, \vec{w} \rangle}{\langle \vec{w}, \vec{w} \rangle}\vec{w}, \vec{w} \right\rangle &= \langle \vec{v}, \vec{w} \rangle - \frac{\langle \vec{v}, \vec{w} \rangle}{\langle \vec{w}, \vec{w} \rangle} \langle \vec{w}, \vec{w} \rangle \\ &= \langle \vec{v}, \vec{w} \rangle - \langle \vec{v}, \vec{w} \rangle \\ &= 0, \end{aligned}$$

as desired. □

You may have come across the orthogonal projection of a vector $\vec{v}$ onto $\vec{w}$ in $\mathbb{R}^n$ before, and the vector is given by $\frac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\vec{w}$. This expression appeared in the previous proof, and we will revisit it later (see Section 4.4). Phrased in this language, Lemma 4.2.4 proves that the perpendicular component of $\vec{v}$ when projected onto $\vec{w}$ is indeed perpendicular to $\vec{w}$.

Theorem 4.2.5 **(Cauchy–Schwarz Inequality)**

Let V be an inner product space. Then

$$|\langle \vec{v}, \vec{w}\rangle| \leq \|\vec{v}\|\,\|\vec{w}\| \qquad \text{for all } \vec{v}, \vec{w} \in V,$$

with equality if and only if $\vec{v}$ and $\vec{w}$ are scalar multiples of each other.

Proof: If $\vec{w} = \vec{0}$ we have $|\langle \vec{v}, \vec{w}\rangle| = \|\vec{v}\|\,\|\vec{w}\| = 0$ so the statement is true. Assume now that $\vec{w} \neq \vec{0}$. We have

$$\begin{aligned}
\|\vec{v}\|^2 &= \left\|\vec{v} - \frac{\langle \vec{v}, \vec{w}\rangle}{\langle \vec{w}, \vec{w}\rangle}\vec{w} + \frac{\langle \vec{v}, \vec{w}\rangle}{\langle \vec{w}, \vec{w}\rangle}\vec{w}\right\|^2 \\
&= \left\|\vec{v} - \frac{\langle \vec{v}, \vec{w}\rangle}{\langle \vec{w}, \vec{w}\rangle}\vec{w}\right\|^2 + \left\|\frac{\langle \vec{v}, \vec{w}\rangle}{\langle \vec{w}, \vec{w}\rangle}\vec{w}\right\|^2 && \text{(by Lemma 4.2.4)} \\
&\geq 0 + \left\|\frac{\langle \vec{v}, \vec{w}\rangle}{\langle \vec{w}, \vec{w}\rangle}\vec{w}\right\|^2 \\
&= \frac{|\langle \vec{v}, \vec{w}\rangle|^2}{\|\vec{w}\|^4}\|\vec{w}\|^2 \\
&= \frac{|\langle \vec{v}, \vec{w}\rangle|^2}{\|\vec{w}\|^2}.
\end{aligned}$$

Since $\|\vec{w}\|^2$ is positive, this implies

$$\|\vec{v}\|^2\,\|\vec{w}\|^2 \geq |\langle \vec{v}, \vec{w}\rangle|^2 .$$

Since norms are always positive real numbers we can take square roots to obtain

$$\|\vec{v}\|\,\|\vec{w}\| \geq |\langle \vec{v}, \vec{w}\rangle| ,$$

giving us the desired inequality. Tracing back our steps, we see that the inequality will be an *equality* if and only if

$$\left\|\vec{v} - \frac{\langle \vec{v}, \vec{w}\rangle}{\langle \vec{w}, \vec{w}\rangle}\vec{w}\right\|^2 = 0$$

which, by Proposition 4.2.7(c) below, is the case if and only if

$$\vec{v} - \frac{\langle \vec{v}, \vec{w}\rangle}{\langle \vec{w}, \vec{w}\rangle}\vec{w} = 0.$$

This completes the proof. □

There is actually a geometric interpretation of the Cauchy–Schwarz inequality, at least in $\mathbb{R}^2$. It turns out that the inequality is a rephrasing of the following fact from geometry: If you have a parallelogram with side lengths x and y, then the area of that parallelogram is maximized exactly when the parallelogram is a rectangle. It's a fun exercise to try and see how this fact relates to the Cauchy–Schwarz inequality for the dot product on $\mathbb{R}^2$!

Because of the Cauchy–Schwarz inequality, we can now sensibly define the angle between two non-zero vectors, at least when our vector space is over the field $\mathbb{R}$.

Definition 4.2.6
Angle

Let V be a real inner product space. The **angle** θ between two non-zero vectors $\vec{v}$ and $\vec{w}$ in V is defined by

$$\cos(\theta) = \frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{v}\| \, \|\vec{w}\|},$$

that is, θ is the unique real number in the interval $[0, \pi]$ given by

$$\theta = \cos^{-1}\left(\frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{v}\| \, \|\vec{w}\|}\right).$$

We do not define the angle between $\vec{v}$ and $\vec{w}$ if one of them is the zero vector.

There are various ways to define the angle between vectors in a complex inner product space, each serving a different purpose. We won't be talking about the angle between vectors in a complex vector space in this course, except for the case when vectors are orthogonal.

We will finish this section by returning to one of our motivations for defining an inner product: a sensible notion of length. The next proposition shows us that our definition of norm provides such a notion.

Proposition 4.2.7 **(Properties of Norm)**

Let V be an inner product space. For all $\vec{v}, \vec{w} \in V$ and $\alpha \in \mathbb{F}$, the following properties are true.

(a) $\|\alpha \vec{v}\| = |\alpha| \, \|\vec{v}\|$.

(b) $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$ (this is called the **Triangle Inequality**).

(c) $\|\vec{v}\| \geq 0$, and $\|\vec{v}\| = 0$ if and only if $\vec{v} = 0$.

Proof: Properties (a) and (c) are left as an exercise. For the triangle inequality we will make use of the Cauchy–Schwarz inequality. We have

$$\begin{aligned}
\|\vec{v} + \vec{w}\|^2 &= \langle \vec{v} + \vec{w}, \vec{v} + \vec{w} \rangle \\
&= \|\vec{v}\|^2 + \|\vec{w}\|^2 + \langle \vec{v}, \vec{w} \rangle + \overline{\langle \vec{v}, \vec{w} \rangle} \\
&= \|\vec{v}\|^2 + \|\vec{w}\|^2 + 2\operatorname{Re}(\langle \vec{v}, \vec{w} \rangle) \\
&\leq \|\vec{v}\|^2 + \|\vec{w}\|^2 + 2\left|\operatorname{Re}(\langle \vec{v}, \vec{w} \rangle)\right| \\
&\leq \|\vec{v}\|^2 + \|\vec{w}\|^2 + 2\left|\langle \vec{v}, \vec{w} \rangle\right| \\
&\leq \|\vec{v}\|^2 + \|\vec{w}\|^2 + 2\|\vec{v}\| \, \|\vec{w}\| \\
&= (\|\vec{v}\| + \|\vec{w}\|)^2.
\end{aligned}$$

Since both sides are positive we have $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$ completing the proof. $\square$

EXERCISE

(a) Prove properties (a) and (c) in Proposition 4.2.7.

(b) Determine when the Triangle Inequality is in fact an *equality*.

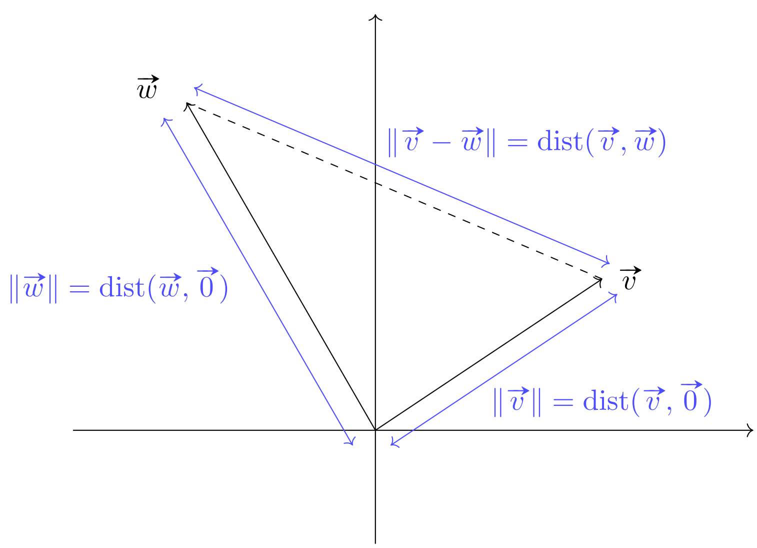
There is an abstract notion of a *norm* on an arbitrary vector space V that is not necessarily an inner product space. It's a function $\| \ \| : V \to \mathbb{F}$ that satisfies the three properties of Proposition 4.2.7. So what we have proved is that what we are calling the norm on an inner product space V is in fact an example of an abstract norm. It is true that there are norms that do not arise this way (i.e. norms that are not built from an inner product), but we will not be concerned with such norms in this course.

Finally, once we have a notion of length, we can speak about the distance between vectors.

Definition 4.2.8
Distance

Let V be an inner product space, and let $\vec{v}, \vec{w} \in V$. The **distance between $\vec{v}$ and $\vec{w}$** is defined as

$$\text{dist}(\vec{v}, \vec{w}) = \|\vec{v} - \vec{w}\|.$$



This definition agrees with our usual notion of distance in $\mathbb{R}^2$ and $\mathbb{R}^3$. Notice that the norm of a vector $\vec{v}$ can now be interpreted to be the distance between $\vec{v}$ and $\vec{0}$: $\|\vec{v}\| = \|\vec{v} - \vec{0}\|$.

Example 4.2.9

In $\mathcal{P}_2(\mathbb{R})$ with $\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)\,dx$, the distance between $1-x$ and $1+x^2$ is given by

$$\begin{aligned}\text{dist}(1-x, 1+x^2) &= \left\|(1-x)-(1+x^2)\right\| \\ &= \left\|-x-x^2\right\| \\ &= |-1|\left\|x+x^2\right\| \\ &= \sqrt{\langle x+x^2, x+x^2\rangle} \\ &= \sqrt{\int_{-1}^{1}(x+x^2)^2\,dx} \\ &= \sqrt{\frac{16}{15}}.\end{aligned}$$

EXERCISE (Properties of Distance)

Let V be an inner product space. Show that the following properties are true for all $\vec{x}, \vec{y}, \vec{z} \in V$.

(a) $\text{dist}(\vec{x}, \vec{y}) \geq 0$, and $\text{dist}(\vec{x}, \vec{y}) = 0$ if and only if $\vec{x} = \vec{y}$.

(b) $\text{dist}(\vec{x}, \vec{y}) = \text{dist}(\vec{y}, \vec{x})$.

(c) $\text{dist}(\vec{x}, \vec{z}) \leq \text{dist}(\vec{x}, \vec{y}) + \text{dist}(\vec{y}, \vec{z})$ (Triangle Inequality).

4.3 Orthonormal Bases

Consider the standard basis in $\mathbb{R}^n$ equipped with the dot product. Each vector in this basis has length 1, and even better, any two vectors in the basis are orthogonal. This will be our gold standard to head towards.

Definition 4.3.1
Orthogonal Set

A set $\{\vec{v}_1, \dots, \vec{v}_k\}$ in an inner product space is called **orthogonal** if $\langle \vec{v}_i, \vec{v}_j\rangle = 0$ whenever $i \neq j$.

Definition 4.3.2
Unit Vector

A vector $\vec{v}$ in an inner product space is a **unit vector** if $\|\vec{v}\| = 1$ (or, equivalently, if $\langle \vec{v}, \vec{v}\rangle = 1$).

Definition 4.3.3
Orthonormal Set

A set $\{\vec{v}_1, \dots, \vec{v}_k\}$ in an inner product space is an **orthonormal set** if it is an orthogonal set and if each vector $\vec{v}_i$ in the set is a unit vector.

Example 4.3.4 Consider $M_{2\times 2}(\mathbb{R})$ with inner product $\langle A, B\rangle = \operatorname{tr}(B^T A)$ from Example 4.1.6.

Define the matrices

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then

$$\langle A, B\rangle = \operatorname{tr}(B^T A) = \operatorname{tr}\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = 0$$
$$\langle A, A\rangle = \operatorname{tr}(A^T A) = \operatorname{tr}\left(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\right) = 1$$
$$\langle B, B\rangle = \operatorname{tr}(B^T B) = \operatorname{tr}\left(\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\right) = 1.$$

Since $\langle A, C\rangle = \langle B, C\rangle = 0$, it follows that $\{A, B, C\}$ is an orthogonal set, but it is not orthonormal (since $\|C\| = 0$). However, $\{A, B\}$ is an orthonormal set.

Example 4.3.5 The set $\left\{\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}\right\}$ is orthonormal in $\mathbb{R}^2$ with respect to the dot product.

Example 4.3.6 **(Legendre polynomials in $\mathcal{P}_3(\mathbb{R})$)**

There are special polynomials called **Legendre polynomials** which arise in physics (specifically when solving Laplace's equation in spherical coordinates), and also in some special trigonometric identities!

The first four Legendre polynomials are the polynomials $\left\{1, x, \frac{3}{2}x^2 - \frac{1}{2}, \frac{5}{2}x^3 - \frac{3}{2}x\right\}$ in $\mathcal{P}_3(\mathbb{R})$. You can check that this is an orthogonal set in $\mathcal{P}_3(\mathbb{R})$ with respect to the inner product

$$\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)\,dx.$$

However, it is not an orthonormal set because

$$\|1\| = \sqrt{\int_{-1}^{1} 1\,dx} = \sqrt{2}$$
$$\|x\| = \sqrt{\int_{-1}^{1} x^2\,dx} = \sqrt{\frac{2}{3}}$$
$$\left\|\frac{3}{2}x^2 - \frac{1}{2}\right\| = \sqrt{\frac{2}{5}}$$
$$\left\|\frac{5}{2}x^3 - \frac{3}{2}x\right\| = \sqrt{\frac{2}{7}}.$$

So none of these Legendre polynomials are unit vectors. If we divide each by its norm, the resulting set

$$\left\{\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}x, \sqrt{\frac{5}{2}}\left(\frac{3}{2}x^2 - \frac{1}{2}\right), \sqrt{\frac{7}{2}}\left(\frac{5}{2}x^3 - \frac{3}{2}x\right)\right\}$$

would then be orthonormal. See the next exercise.

EXERCISE (Normalization)

Let $\vec{v}$ be a non-zero vector in an inner product space. The **normalization** of $\vec{v}$ is the vector $\widehat{v} = \dfrac{\vec{v}}{\|\vec{v}\|}$.

(a) Prove that $\widehat{v}$ is a unit vector.

(b) Suppose that $S = \{\vec{v}_1, \dots, \vec{v}_k\}$ is an orthogonal set. Prove that $T = \{\widehat{v}_1, \cdots, \widehat{v}_k\}$ is an orthonormal set.

If our model for an orthogonal set is the standard basis in $\mathbb{R}^n$, we should expect an orthogonal set to be linearly independent. Indeed that is the case—at least if none of the vectors is the zero vector.

Proposition 4.3.7 Suppose $\{\vec{v}_1, \dots, \vec{v}_k\}$ is orthogonal and $\vec{v}_i \neq \vec{0}$ for all i. Then $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly independent.

Proof: Suppose $t_1\vec{v}_1 + \cdots + t_k\vec{v}_k = \vec{0}$. Fix $i \in \{1, \dots, k\}$. Then

$$\begin{aligned} 0 &= \langle t_1\vec{v}_1 + \cdots + t_k\vec{v}_k, \vec{v}_i\rangle \\ &= t_1 \langle \vec{v}_1, \vec{v}_i\rangle + \cdots + t_i \langle \vec{v}_i, \vec{v}_i\rangle + \cdots + t_k \langle \vec{v}_k, \vec{v}_i\rangle \\ &= t_i \|\vec{v}_i\|^2 . \end{aligned}$$

Since $\vec{v}_i \neq \vec{0}$, we must have $t_i = 0$. Since this is true for all i, we conclude $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly independent. □

We now make a special definition for the case that we have an orthonormal set which forms a basis for an inner product space.

Definition 4.3.8
Orthonormal Basis

A set $\{\vec{v}_1, \dots, \vec{v}_n\}$ in an inner product space V is an **orthonormal basis** if it is an orthonormal set and it is a basis for V.

Example 4.3.9 The following are orthonormal bases, as you can easily check.

1. The standard basis of $\mathbb{R}^n$ with respect to the dot product.
2. The standard basis of $\mathbb{C}^n$ with respect to the standard inner product on $\mathbb{C}^n$.
3. The standard basis of $M_{m\times n}(\mathbb{R})$ with respect to the inner product $\langle A, B\rangle = \mathrm{tr}(B^T A)$.
4. The standard basis of $M_{m\times n}(\mathbb{C})$ with respect to the inner product $\langle A, B\rangle = \mathrm{tr}(B^* A)$.

Example 4.3.10 The standard basis of $\mathcal{P}_n(\mathbb{R})$ with respect to the inner product

$$\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)\,dx$$

is **not** an orthonormal basis. In fact, it is not even an orthogonal set if $n > 1$. (See Example 4.2.2.)

In Example 4.3.6 we showed that the set

$$\left\{\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}x, \sqrt{\frac{5}{2}}\left(\frac{3}{2}x^2 - \frac{1}{2}\right), \sqrt{\frac{7}{2}}\left(\frac{5}{2}x^3 - \frac{3}{2}x\right)\right\}$$

is an orthonormal basis for $\mathcal{P}_3(\mathbb{R})$ with respect to the above inner product. In Section 4.5 we'll see how to create an orthonormal basis for $\mathcal{P}_n(\mathbb{R})$ for all $n \geq 1$.

Example 4.3.11 In $\mathbb{C}^2$, with the inner product

$$\left\langle \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \right\rangle = 2a\overline{c} + b\overline{d},$$

the set $\left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is an orthonormal basis, as you can check.

EXERCISE

Check that the sets given in Examples 4.3.9 and 4.3.11 are orthonormal bases.

We close this section by giving a result that might explain why we sometimes prefer to work with an orthonormal (or even orthogonal) basis: it makes it very easy to find the coordinates of any given vector.

Proposition 4.3.12 **(Coordinates Relative to an Orthogonal Basis)**

Let V be an inner product space and let $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis for V. If $\vec{x} \in V$ is given by $\vec{x} = x_1\vec{v}_1 + \cdots + x_n\vec{v}_n$, then:

(a) If $\mathcal{B}$ is an orthogonal basis, then $x_i = \dfrac{\langle \vec{x}, \vec{v}_i\rangle}{\langle \vec{v}_i, \vec{v}_i\rangle}$ for $1 \leq i \leq n$.

(b) If $\mathcal{B}$ is an orthonormal basis, then $x_i = \langle \vec{x}, \vec{v}_i\rangle$ for $1 \leq i \leq n$.

Proof: If $\mathcal{B}$ is orthogonal, then by taking the inner product of $\vec{x}$ and $\vec{v}_i$ we obtain

$$\begin{aligned}\langle \vec{x}, \vec{v}_i \rangle &= \langle x_1 \vec{v}_1 + \cdots + x_n \vec{v}_n, \vec{v}_i \rangle \\ &= x_1 \langle \vec{v}_1, \vec{v}_i \rangle + \cdots + x_i \langle \vec{v}_i, \vec{v}_i \rangle + \cdots + x_n \langle \vec{v}_n, \vec{v}_i \rangle \\ &= x_1 0 + \cdots + x_{i-1} 0 + x_i \langle \vec{v}_i, \vec{v}_i \rangle + x_{i+1} 0 + \cdots + x_n 0 \\ &= x_i \langle \vec{v}_i, \vec{v}_i \rangle .\end{aligned}$$

Now since $\langle \vec{v}_i, \vec{v}_i \rangle \neq 0$ (why?), we can divide through by $\langle \vec{v}_i, \vec{v}_i \rangle$ to obtain

$$x_i = \frac{\langle \vec{x}, \vec{v}_i \rangle}{\langle \vec{v}_i, \vec{v}_i \rangle}.$$

This proves (a). Part (b) follows immediately since if $\mathcal{B}$ is orthonormal then $\langle \vec{v}_i, \vec{v}_i \rangle = 1$ for all i. □

Example 4.3.13 In $\mathbb{R}^n$ with the standard basis $\mathcal{S} = \{\vec{e}_1, \ldots, \vec{e}_n\}$ (which is orthonormal with respect to the dot product), the above proposition says that we can obtain the i^{th} component of $\vec{x} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^T$ as $\vec{x} \cdot \vec{e}_i$. And indeed we can:

$$\vec{x} \cdot \vec{e}_i = \begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = x_i.$$

Example 4.3.14 In $M_{2\times 2}(\mathbb{R})$, let $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ 2 & -3 \end{bmatrix} \right\}$ and $W = \text{Span}(\mathcal{B})$. You can check that $\mathcal{B}$ is an orthogonal basis for W with respect to the inner product $\langle A, B \rangle = \text{tr}(B^T A)$. Supposing we know that $A = \begin{bmatrix} 3 & -2 \\ -1 & -4 \end{bmatrix}$ is in W, we can find its coordinate vector $[A]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ quickly as follows. We have

$$a_1 = \frac{\left\langle A, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\rangle}{\left\| \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\|^2} = \frac{3(1) + (-2)(1) + (-1)(1) + (-4)(1)}{1^2 + 1^2 + 1^2 + 1^2} = -1$$

$$a_2 = \frac{\left\langle A, \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \right\rangle}{\left\| \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \right\|^2} = \frac{3(1) + (-2)(0) + (-1)(-1) + (-4)(0)}{1^2 + (-1)^2 + 0^2 + 0^2} = 2$$

$$a_3 = \frac{\left\langle A, \begin{bmatrix} 2 & -1 \\ 2 & -3 \end{bmatrix} \right\rangle}{\left\| \begin{bmatrix} 2 & -1 \\ 2 & -3 \end{bmatrix} \right\|^2} = \frac{3(2) + (-2)(-1) + (-1)(2) + (-4)(-3)}{2^2 + (-1)^2 + 2^2 + (-3)^2} = 1.$$

Thus, $[A]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$.

Our goal now is to show that *every* finite-dimensional inner product space admits an orthonormal basis. As a fun exercise, try to see if you can prove the following special case.

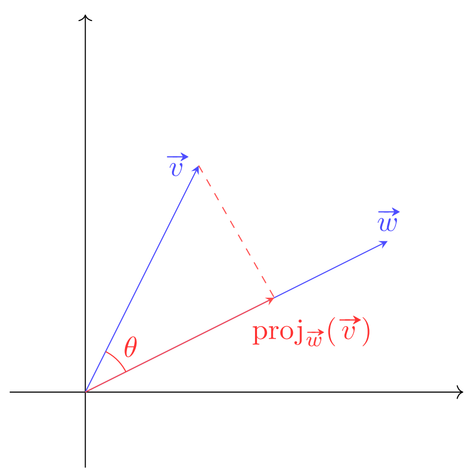
EXERCISE

Let V be an inner product space and let $W = \text{Span}(\{\vec{v}, \vec{w}\})$, where $\{\vec{v}, \vec{w}\}$ is a linearly independent subset of V. Obtain an orthonormal basis for W. [**Hint:** Start by trying to find a vector $a\vec{v} + b\vec{w}$ in W that is orthogonal to $\vec{v}$.]

4.4 Projections

You may have seen the projection of a vector onto another vector in a previous course, and we have seen hints of it in the proof of the Cauchy–Schwarz inequality. We now shift our attention to fleshing out the details in full.

Let's think about what projection looks like in $\mathbb{R}^2$. Suppose $\vec{v}$ and $\vec{w}$ are two non-zero vectors in $\mathbb{R}^2$, and we wish to project $\vec{v}$ onto $\vec{w}$. We can think of this as shining a light perpendicular to $\vec{w}$, and drawing a vector $\text{proj}_{\vec{w}}(\vec{v})$ representing the shadow of $\vec{v}$.



Suppose θ is the angle between $\vec{v}$ and $\vec{w}$. Then by drawing out a right triangle, we see the length of the projection must be $\|\vec{v}\| \cos\theta$. The direction we wish the vector to go in is the direction $\vec{w}$ is pointing, so we can obtain the projection by scalar multiplying the unit vector in the direction of $\vec{w}$ by $\|\vec{v}\| \cos\theta$. Since $\cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|}$, this gives the projection as

$$\text{proj}_{\vec{w}}(\vec{v}) = \|\vec{v}\| \cos\theta \frac{1}{\|\vec{w}\|}\vec{w} = \frac{\vec{v}\cdot\vec{w}}{\|\vec{w}\|^2}\vec{w}.$$

We will use this as motivation for defining projections in inner product spaces in general.

Definition 4.4.1

Projection onto a Vector, $\text{proj}_{\vec{w}}$, Perpendicular Vector with respect to a Vector, $\text{perp}_{\vec{w}}$

Let V be an inner product space, and let $\vec{w}, \vec{v} \in V$ with $\vec{w} \neq \vec{0}$. The **projection of $\vec{v}$ onto $\vec{w}$** is defined to be the vector

$$\text{proj}_{\vec{w}}(\vec{v}) = \frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{w}\|^2}\vec{w}.$$

We also define the **perpendicular vector of $\vec{v}$ with respect to $\vec{w}$** by

$$\text{perp}_{\vec{w}}(\vec{v}) = \vec{v} - \frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{w}\|^2}\vec{w}.$$

We already saw in Lemma 4.2.4 that $\text{perp}_{\vec{w}}(\vec{v})$ is orthogonal to $\vec{w}$, which is what we'd expect to be true if these definitions imitate the situation in $\mathbb{R}^2$. Furthermore, notice that according to the definition

$$\vec{v} = \text{proj}_{\vec{w}}(\vec{v}) + \text{perp}_{\vec{w}}(\vec{v})$$

for all $\vec{v}, \vec{w} \in V$ with $\vec{w} \neq \vec{0}$.

Example 4.4.2

Let $\vec{v} = \begin{bmatrix} 4 \\ 1+i \\ 2 \end{bmatrix} \in \mathbb{C}^3$. Let $\vec{w}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. With respect to the standard inner product we have

$$\text{proj}_{\vec{w}_1}(\vec{v}) = \frac{2}{1}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$\text{proj}_{\vec{w}_2}(\vec{v}) = \frac{i+1}{1}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ i+1 \\ 0 \end{bmatrix}$$

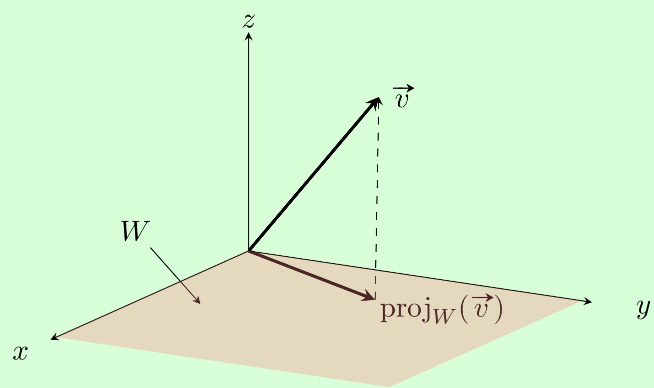
These computations are what we'd expect if we are to project onto the z and y axes of $\mathbb{C}^3$.

Another way to think about a projection is as finding the closest vector on a subspace to a given vector. More specifically, the projection of $\vec{v}$ onto $\vec{w}$ is the closest vector in the subspace spanned by $\{\vec{w}\}$ to $\vec{v}$. (See Proposition 4.6.8 below.) Remember—since we're in an inner product space, we have a notion of length and distance, so asking for the *closest* vector makes sense.

With this in mind, what we're really doing when we're projecting onto a vector is we're projecting onto the one-dimensional subspace spanned by that vector. It's natural to now ask how we can project onto a general subspace. Let's look at an example.

Example 4.4.3

Consider $\mathbb{R}^3$ with the dot product. Let $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. The subspace $W = \text{Span}(\{\vec{e}_1, \vec{e}_2\})$ is the xy-plane in $\mathbb{R}^3$. So the projection of $\vec{v}$ onto W, let's call it $\text{proj}_W(\vec{v})$, should be the vector $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.



If we compute the projections of $\vec{v}$ onto $\vec{e}_1$ and $\vec{e}_2$, we find that

$$\mathrm{proj}_{\vec{e}_1}(\vec{v}) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathrm{proj}_{\vec{e}_2}(\vec{v}) = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}.$$

The sum of these two vector projections is what we believe $\mathrm{proj}_W(\vec{v})$ should be. That is, it appears that

$$\mathrm{proj}_W(\vec{v}) = \mathrm{proj}_{\vec{e}_1}(\vec{v}) + \mathrm{proj}_{\vec{e}_2}(\vec{v}).$$

From this example, it may be tempting to guess the following: The projection of a vector $\vec{v}$ onto a subspace W is simply obtained by choosing a basis for W, projecting $\vec{v}$ onto each basis vector, and summing up the resulting vectors.

Unfortunately this does not work all the time, but we will see that it it does work when we have an orthogonal basis for W. So it may be tempting to refine our initial guess and define the projection of $\vec{v}$ onto W as follows. Let $\{\vec{w}_1, \ldots, \vec{w}_k\}$ be an orthogonal basis for W. Then

$$\mathrm{proj}_W(\vec{v}) = \mathrm{proj}_{\vec{w}_1}(\vec{v}) + \cdots + \mathrm{proj}_{\vec{w}_k}(\vec{v}).$$

For this to be a usable definition, there are two issues that must be addressed.

1. Does W even have an orthogonal basis? If so, how do we find one?
2. Does the above definition of $\mathrm{proj}_W(\vec{v})$ depend on the chosen orthogonal basis for W? That is, if $\{\vec{u}_1, \ldots, \vec{u}_k\}$ is another orthogonal basis for W, how can we be sure that

$$\mathrm{proj}_{\vec{w}_1}(\vec{v}) + \cdots + \mathrm{proj}_{\vec{w}_k}(\vec{v}) = \mathrm{proj}_{\vec{u}_1}(\vec{v}) + \cdots + \mathrm{proj}_{\vec{u}_k}(\vec{v})$$

   so that our definition of $\mathrm{proj}_W(\vec{v})$ is well-defined?

We will address both these issues in the next two sections.

4.5 The Gram–Schmidt Orthogonalization Procedure

In this section we will show that every finite-dimensional inner product space has an orthonormal basis. In fact, we will describe a procedure that allows us to take an arbitrary basis and create an orthonormal basis from it. Let's see how this works in an example.

Example 4.5.1 In $\mathbb{R}^3$ equipped with the dot product, suppose we have

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

It turns out that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$. However, $\vec{v}_1 \cdot \vec{v}_2 = 1$ so it is not an orthogonal set (or an orthonormal one for that matter). We will now create an orthogonal basis starting form this one, and then scale the vectors to obtain an orthonormal basis.

Let's first create an orthogonal set $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$. We may as well start with

$$\vec{w}_1 = \vec{v}_1.$$

Now, whatever we make $\vec{w}_2$, it needs to be orthogonal to $\vec{w}_1$. We have proved earlier that $\operatorname{perp}_{\vec{w}_1}(\vec{v}_2)$ is orthogonal to $\vec{w}_1$, so let's use that. We have

$$\begin{aligned} \operatorname{perp}_{\vec{w}_1}(\vec{v}_2) &= \vec{v}_2 - \frac{\langle \vec{v}_2, \vec{w}_1 \rangle}{\|\vec{w}_1\|^2} \vec{w}_1 \\ &= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}. \end{aligned}$$

So that we don't have to deal with fractions, set

$$\vec{w}_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

So far we've obtained two orthogonal vectors $\vec{w}_1$ and $\vec{w}_2$. To create $\vec{w}_3$, we do the same thing except we will want to take

$$\operatorname{perp}_W(\vec{v}_3) = \vec{v}_3 - \operatorname{proj}_W(\vec{v}_3), \quad \text{where } W = \operatorname{Span}(\{\vec{w}_1, \vec{w}_2\}),$$

which will give us a vector orthogonal to both $\vec{w}_1$ and $\vec{w}_2$. Of course, the problem here is that we haven't formally defined $\operatorname{perp}_W(\vec{v}_3)$ or $\operatorname{proj}_W(\vec{v}_3)$, but based on Example 4.4.3 and the discussion that followed it, we suspect that

$$\operatorname{perp}_W(\vec{v}_3) = \vec{v}_3 - \operatorname{proj}_{\vec{w}_1} \vec{w}_1 - \operatorname{proj}_{\vec{w}_2} \vec{w}_2.$$

Let's just take $\vec{w}_3$ to be this vector! That is, let

$$\begin{aligned}\vec{w}_3 &= \vec{v}_3 - \operatorname{proj}_{\vec{w}_1} \vec{v}_3 - \operatorname{proj}_{\vec{w}_2} \vec{v}_3 \\ &= \vec{v}_3 - \frac{\langle \vec{v}_3, \vec{w}_1 \rangle}{\|\vec{w}_1\|^2}\vec{w}_1 - \frac{\langle \vec{v}_3, \vec{w}_2 \rangle}{\|\vec{w}_2\|^2}\vec{w}_2 \\ &= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}.\end{aligned}$$

Again, to make our lives easier, set

$$\vec{w}_3 = \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}.$$

We have to confirm that $\vec{w}_3$ behaves as expected and is in fact orthogonal to both $\vec{w}_1$ and $\vec{w}_2$. Indeed, this is the case! We have

$$\vec{w}_1 \cdot \vec{w}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix} = -2 + 2 = 0$$

and

$$\vec{w}_1 \cdot \vec{w}_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix} = -2 - 2 + 3 = 0.$$

Now we have three vectors that are orthogonal to each other, so $\mathcal{B} = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is an orthogonal set. Since $\vec{0} \notin \mathcal{B}$, it follows that $\mathcal{B}$ linearly independent and thus a basis for $\mathbb{R}^3$.

To create an orthonormal basis for $\mathbb{R}^3$ we simply normalize the vectors in S. Therefore

$$\mathcal{C} = \left\{ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is an orthonormal basis of $\mathbb{R}^3$ with respect to the dot product.

The method we illustrated in this example works in general, and it is called the **Gram–Schmidt orthogonalization procedure**. Here it is in detail.

ALGORITHM (Gram–Schmidt Orthogonalization Procedure)

Let V be an inner product space with basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$. To obtain an orthogonal basis for V, define $\vec{w}_1, \ldots, \vec{w}_n$ as follows.

$$\begin{aligned}
\vec{w}_1 &= \vec{v}_1 \\
\vec{w}_2 &= \vec{v}_2 - \frac{\langle \vec{v}_2, \vec{w}_1 \rangle}{\|\vec{w}_1\|^2}\vec{w}_1 \\
\vec{w}_3 &= \vec{v}_3 - \frac{\langle \vec{v}_3, \vec{w}_1 \rangle}{\|\vec{w}_1\|^2}\vec{w}_1 - \frac{\langle \vec{v}_3, \vec{w}_2 \rangle}{\|\vec{w}_2\|^2}\vec{w}_2 \\
&\vdots \\
\vec{w}_n &= \vec{v}_n - \frac{\langle \vec{v}_n, \vec{w}_1 \rangle}{\|\vec{w}_1\|^2}\vec{w}_1 - \cdots - \frac{\langle \vec{v}_n, \vec{w}_{n-1} \rangle}{\|\vec{w}_{n-1}\|^2}\vec{w}_{n-1}.
\end{aligned}$$

Then $\mathcal{B} = \{\vec{w}_1, \ldots, \vec{w}_n\}$ is an orthogonal basis for V.

To obtain an orthonormal basis, let $\vec{u}_i = \widehat{w_i} = \frac{1}{\|\vec{w}_i\|}\vec{w}_i$, and take $\mathcal{C} = \{\vec{u}_1, \ldots, \vec{u}_n\}$.

To prove that this procedure actually works, we need to ensure that at each step we actually get a non-zero vector. This amounts to proving the following statement.

EXERCISE

Let $\{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis for an inner product space V. Let $\{\vec{w}_1, \ldots, \vec{w}_i\}$ be the first i vectors obtained from the Gram–Schmidt orthogonalization procedure. Prove $\text{Span}(\{\vec{v}_1, \ldots, \vec{v}_i\}) = \text{Span}(\{\vec{w}_1, \ldots, \vec{w}_i\})$. Use this to prove that the resulting basis $\{\vec{w}_1, \ldots, \vec{w}_n\}$ is an orthogonal basis for V.

As a consequence of the Gram–Schmidt procedure, we get the following corollary.

Corollary 4.5.2 Every finite-dimensional inner product space admits an orthonormal basis.

If $V = \{\vec{0}\}$ is the zero vector space, then we agree to consider its basis (the empty set) as being an orthogonal basis. This makes sense because the condition for the empty set to be an orthogonal set is vacuously true.

Example 4.5.3 Let's apply the Gram–Schmidt process to find an orthogonal basis for $P_3(\mathbb{R})$ with respect to the inner product $\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)\,dx$. Starting from the standard basis $\{1, x, x^2, x^3\}$, we take $\vec{w}_1 = 1$, and then

$$\vec{w}_2 = x - \frac{\langle x, 1 \rangle}{\|1\|^2}1 = x - \frac{0}{2}1 = x.$$

(Recall that we'd already seen that $1 \perp x$, so this is a promising start.) Next,

$$\vec{w}_3 = x^2 - \frac{\langle x^2, 1 \rangle}{\|1\|^2}1 - \frac{\langle x^2, x \rangle}{\|x\|^2}x = x^2 - \frac{\frac{2}{3}}{2}1 - \frac{0}{\frac{2}{3}}x = x^2 - \frac{1}{3}.$$

To avoid having to deal with fractions, let's multiply by 3 and take $\vec{w}_3 = 3x^2 - 1$ instead. Finally,

$$\vec{w}_4 = x^3 - \frac{\langle x^3, 1\rangle}{\|1\|^2}1 - \frac{\langle x^3, x\rangle}{\|x\|^2}x - \frac{\langle x^3, 3x^2-1\rangle}{\|3x^2-1\|^2}(3x^2-1) = x^3 - \frac{3}{5}x.$$

Let's multiply by 5 and take $\vec{w}_4 = 5x^3 - 3x$ instead.

So now we have an orthogonal basis $\{1, x, 3x^2-1, 5x^3-3x\}$ for $\mathcal{P}_3(\mathbb{R})$ with respect to $\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)\,dx$. If we normalize this, we obtain the orthonormal basis

$$\mathcal{C} = \left\{\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}x, \sqrt{\frac{5}{8}}\left(3x^2-1\right), \sqrt{\frac{7}{8}}\left(5x^3-3x\right)\right\}$$

which is exactly the same as the basis we'd seen in Example 4.3.6.

EXERCISE

Find an orthonormal basis for $\mathcal{P}_4(\mathbb{R})$ with respect to $\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)\,dx$.

Example 4.5.4 Let's find an orthogonal basis for the subspace

$$W = \text{Span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$$

of $\mathbb{R}^3$ with respect to the dot product.

Our first step should be to find a basis for W, which means we must eliminate any redundant vectors in the given spanning set. (We know for sure that at least one of them must be redundant, since W cannot be a 4-dimensional subspace of $\mathbb{R}^3$!) But let's *not* do this, and let's see what happens if we apply the Gram–Schmidt procedure to the given spanning set of W.

So take $\vec{w}_1 = \begin{bmatrix}1 & 1 & 1\end{bmatrix}^T$, and then

$$\vec{w}_2 = \begin{bmatrix}1\\-1\\0\end{bmatrix} - \frac{\begin{bmatrix}1\\-1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}{\left\|\begin{bmatrix}1\\1\\1\end{bmatrix}\right\|^2}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\-1\\0\end{bmatrix}.$$

Next, we are supposed to take

$$\vec{w}_3 = \begin{bmatrix}2\\0\\1\end{bmatrix} - \frac{\begin{bmatrix}2\\0\\1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}}{\left\|\begin{bmatrix}1\\1\\1\end{bmatrix}\right\|^2}\begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{\begin{bmatrix}2\\0\\1\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\end{bmatrix}}{\left\|\begin{bmatrix}1\\-1\\0\end{bmatrix}\right\|^2}\begin{bmatrix}1\\-1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix},$$

but we cannot have the zero vector in a basis! Did the Gram–Schmidt procedure fail? No, of course not. In fact, the procedure was able to detect the linear dependence present in the spanning set of W. Indeed, the above shows that the vector $\begin{bmatrix}2 & 0 & 1\end{bmatrix}^T$ is a linear combination of $\begin{bmatrix}1 & 1 & 1\end{bmatrix}^T$ and $\begin{bmatrix}1 & -1 & 0\end{bmatrix}^T$. So that means we toss it aside and move on to the next vector in the spanning set. Hence, we now take

$$\vec{w}_3 = \begin{bmatrix}1\\0\\1\end{bmatrix} - \frac{\begin{bmatrix}1\\0\\1\end{bmatrix} \cdot \begin{bmatrix}1\\1\\1\end{bmatrix}}{\left\|\begin{bmatrix}1\\1\\1\end{bmatrix}\right\|^2}\begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{\begin{bmatrix}1\\0\\1\end{bmatrix} \cdot \begin{bmatrix}1\\-1\\0\end{bmatrix}}{\left\|\begin{bmatrix}1\\-1\\0\end{bmatrix}\right\|^2}\begin{bmatrix}1\\-1\\0\end{bmatrix} = \frac{1}{6}\begin{bmatrix}-1\\-1\\2\end{bmatrix}.$$

We can scale this vector by 6 and then arrive at the following orthogonal basis for W:

$$\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}-1\\-1\\2\end{bmatrix}\right\}.$$

What occurred in the previous example will always occur. The Gram–Schmidt process will automatically locate linear dependency in a given spanning set. This is formalized in the next exercise.

EXERCISE

Let $S = \{\vec{v}_1, \ldots, \vec{v}_k\}$ be linearly independent vectors in an inner product space, and let $\vec{w}_1, \ldots, \vec{w}_k$ be the vectors produced by the Gram–Schmidt procedure applied to S. Show that if $\vec{v}_{k+1} \in \text{Span}(S)$, and if we take

$$\vec{w}_{k+1} = \vec{v}_{k+1} - \frac{\langle \vec{v}_{k+1}, \vec{w}_1 \rangle}{\|\vec{w}_1\|^2}\vec{w}_1 - \cdots - \frac{\langle \vec{v}_{k+1}, \vec{w}_k \rangle}{\|\vec{w}_k\|^2}\vec{w}_k$$

then necessarily $\vec{w}_{k+1} = 0$.

4.6 Orthogonal Complements

Every subspace W of an inner product space V determines a "complementary" subspace that consists of all the vectors in V that are orthogonal to every vector in W.

Definition 4.6.1
Orthogonal Complement

Let V be an inner product space and let $W \subseteq V$ be a subspace. The **orthogonal complement** of W is the set

$$W^\perp = \{\vec{v} \in V : \langle \vec{v}, \vec{w} \rangle = 0 \text{ for all } \vec{w} \in W\}.$$

Here are two basic properties of $W^\perp$ whose proofs are left as an exercise.

Proposition 4.6.2 Let V be an inner product space and $W \subseteq V$ a subspace. Then:

(a) $W^\perp$ is a subspace of V.

(b) $W \cap W^\perp = \{\vec{0}\}$.

EXERCISE

Prove Proposition 4.6.2.

Example 4.6.3 Consider $\mathbb{R}^3$ with the dot product and let $W = \text{Span}\left(\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}\right)$. We would expect $U = \text{Span}\left(\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}\right)$ to be equal to $W^\perp$. Let's prove this.

Proof: We want to show $U = W^\perp$. Let $\vec{u} \in U$. Then $\vec{u} = \begin{bmatrix}0\\0\\t\end{bmatrix}$ for some $t \in \mathbb{R}$. Let $\vec{w}$ be an arbitrary vector in W, so $\vec{w} = \begin{bmatrix}a\\b\\0\end{bmatrix}$ for some $a, b \in \mathbb{R}$. Then $\vec{u} \cdot \vec{w} = 0$ so $\vec{u} \in W^\perp$ and therefore $U \subseteq W^\perp$. Conversely, suppose $\vec{u} = \begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix} \in W^\perp$. Then since $\vec{u} \cdot \begin{bmatrix}1\\0\\0\end{bmatrix} = 0$, we must have $u_1 = 0$. Similarly we must have $u_2 = 0$. Therefore $\vec{u} \in U$ and $W^\perp \subseteq U$. We can now conclude $W^\perp = U$, completing the proof. $\square$

Example 4.6.4 We saw earlier that

$$\left\{1, x, \frac{3}{2}x^2 - \frac{1}{2}, \frac{5}{2}x^3 - \frac{3}{2}x\right\}$$

is an orthogonal basis for $\mathcal{P}_3(\mathbb{R})$ with respect to the inner product $\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)\,dx$. Therefore if $W = \text{Span}\left(\left\{1, \frac{3}{2}x^2 - \frac{1}{2}\right\}\right)$ then $W^\perp = \text{Span}\left(\left\{x, \frac{5}{2}x^3 - \frac{3}{2}x\right\}\right)$. We can prove this as we did in the previous example.

However, if we stop and think for a moment, we might suspect that there is a general result along the lines of: If $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is an orthogonal basis for V, and if $W = \text{Span}\{\vec{v}_1, \ldots, \vec{v}_k\}$, then $W^\perp = \{\vec{v}_{k+1}, \ldots, \vec{v}_n\}$. This is indeed true and the proof conceptually is identical to what we did in the previous example.

EXERCISE

Prove the claim in the final paragraph of Example 4.6.4.

There is a sort-of converse to this exercise. We record it together with a few other basic properties of the orthogonal complement in the next proposition.

Proposition 4.6.5 Let V be a finite-dimensional inner product space and $W \subseteq V$ a subspace. Then:

(a) If $\{\vec{w}_1, \ldots, \vec{w}_k\}$ is a spanning set for W then $\vec{v} \in W^{\perp}$ if and only if $\langle \vec{v}, \vec{w}_i \rangle = 0$ for all $i = 1, \ldots, k$.

(b) If $\mathcal{B} = \{\vec{w}_1, \ldots, \vec{w}_k\}$ is an orthogonal basis for W, then there exists an orthogonal basis $\mathcal{C} = \{\vec{v}_1, \ldots, \vec{v}_m\}$ for $W^{\perp}$ such that $\mathcal{B} \cup \mathcal{C} = \{\vec{w}_1, \ldots, \vec{w}_k, \vec{v}_1, \ldots, \vec{v}_m\}$ is an orthogonal basis for V.

(c) $\dim(V) = \dim(W) + \dim(W^{\perp})$.

(d) $(W^{\perp})^{\perp} = W$.

Proof: (a) If $\vec{v} \in W^{\perp}$, then $\langle \vec{v}, \vec{w} \rangle = 0$ for all $\vec{w} \in W$, so in particular for all $\vec{w}_i$. Conversely, assume that $\langle \vec{v}, \vec{w}_i \rangle = 0$ for all $\vec{w}_i$. To show that $\vec{v} \in W^{\perp}$, we must prove that $\langle \vec{v}, \vec{w} \rangle = 0$ for all $\vec{w} \in W$. Given $\vec{w} \in W$, we can write it as $\vec{w} = \sum_{i=1}^{k} a_i \vec{w}_i$. Then

$$\langle \vec{v}, \vec{w} \rangle = \left\langle \vec{v}, \sum_{i=1}^{k} a_i \vec{w}_i \right\rangle = \sum_{i=1}^{k} \overline{a_i} \langle \vec{v}, \vec{w}_i \rangle = \sum_{i=1}^{k} \overline{a_i} 0 = 0,$$

as desired.

(b) First extend the basis $\mathcal{B}$ for W to a basis $\{\vec{w}_1, \ldots, \vec{w}_k, \vec{x}_1, \ldots, \vec{x}_m\}$ for V and then apply Gram–Schmidt procedure to this basis.

The result will be a basis $\{\vec{w}_1, \ldots, \vec{w}_k, \vec{v}_1, \ldots, \vec{v}_m\}$ for V whose first k vectors are the vectors in $\mathcal{B}$, since $\mathcal{B}$ is orthogonal, and then the remaining vectors $\vec{v}_1, \ldots, \vec{v}_m$ will be orthogonal to all the vectors in $\mathcal{B}$ and thus must lie in $W^{\perp}$ by part (a). We need to show that $\mathcal{C} = \{\vec{v}_1, \ldots, \vec{v}_m\}$ is really a basis for $W^{\perp}$. Since $\mathcal{C}$ is linearly independent (being a subset of a basis), it suffices to show that $\mathcal{C}$ spans $W^{\perp}$.

So suppose $\vec{v} \in W^{\perp}$. Then since $\mathcal{B} \cup \mathcal{C}$ is a basis for V, we can write $\vec{v}$ as

$$\vec{v} = \sum_{i=1}^{k} a_i \vec{w}_i + \sum_{j=1}^{m} b_j \vec{v}_j.$$

Taking inner product of both sides with $\vec{w}_l$, we obtain

$$\langle \vec{v}, \vec{w}_l \rangle = \sum_{i=1}^{k} a_i \langle \vec{w}_i, \vec{w}_l \rangle + \sum_{j=1}^{m} b_j \langle \vec{v}_j, \vec{w}_l \rangle$$

$$0 = a_l \langle \vec{w}_l, \vec{w}_l \rangle + \sum_{j=1}^{m} b_j 0$$

since $\vec{v}, \vec{v}_j \in W^\perp$ for all j and since $\vec{w}_i \perp \vec{w}_l$ for $i \neq l$ since $\mathcal{B}$ is orthogonal. This leaves us with $a_l \langle \vec{w}_l, \vec{w}_l \rangle$ which shows that $a_l = 0$ for all l (since $\vec{w}_l \neq \vec{0}$), hence

$$\vec{v} = \sum_{j=1}^{m} b_j \vec{v}_j.$$

This proves that $\mathcal{C}$ is a spanning set of $W^\perp$, as desired.

(c) This follows from part (b).

(d) If we apply part (c) to the subspace $W^\perp$ (instead of W), we obtain

$$\dim(V) = \dim(W^\perp) + \dim((W^\perp)^\perp).$$

On the other hand, by part (c) applied to W, we have

$$\dim(V) = \dim(W) + \dim(W^\perp).$$

Equating both expressions, we obtain $\dim(W) = \dim((W^\perp)^\perp)$. However, W is a subspace of $(W^\perp)^\perp$, since all the vectors in W are orthogonal to all the vectors in $W^\perp$. This forces $W = (W^\perp)^\perp$ by Theorem 1.3.14.

□

REMARK

Part (a) of the previous proposition remains true if V were infinite-dimensional, but the remaining parts fail. While it is always true that $W \subseteq (W^\perp)^\perp$, there are examples where $W \neq (W^\perp)^\perp$ in an infinite-dimensional inner product space.

4.6.1 Projection onto a Subspace

We're going to show that every vector $\vec{v} \in V$ can be decomposed as the sum of a vector in W and a vector in $W^\perp$. The idea here is that we can write $\vec{v}$ as the sum of $\text{proj}_W(\vec{v})$ and $\text{perp}_W(\vec{v})$, but we don't want to use this terminology just yet (since we're going to use this result to prove that our definitions of proj_W and perp_W are well-defined).

Theorem 4.6.6 **(Orthogonal Decomposition)**

Let V be a finite-dimensional inner product space and $W \subseteq V$ a subspace. Then every $\vec{v} \in V$ can be written as $\vec{v} = \vec{p} + \vec{r}$ where $\vec{p} \in W$ and $\vec{r} \in W^\perp$ are uniquely determined by $\vec{v}$.

Moreover, if $\mathcal{B} = \{\vec{w}_1, \ldots, \vec{w}_k\}$ is an orthogonal basis for W, then $\vec{p}$ is given by

$$\vec{p} = \sum_{i=1}^{k} \frac{\langle \vec{v}, \vec{w}_i \rangle}{\|\vec{w}_i\|^2} \vec{w}_i.$$

Proof: Let $\mathcal{B} = \{\vec{w}_1, \ldots, \vec{w}_k\}$ be an orthogonal basis for W, and extend it to an orthogonal basis $\{\vec{w}_1, \ldots, \vec{w}_k, \vec{v}_1, \ldots, \vec{v}_m\}$ for V, where $\mathcal{C} = \{\vec{v}_1, \ldots, \vec{v}_m\}$ is a basis for $W^\perp$. (We can do this thanks to Proposition 4.6.5(b).) Then we can write $\vec{v}$ as

$$\vec{v} = \sum_{i=1}^{k} a_i \vec{w}_i + \sum_{j=1}^{m} b_j \vec{v}_j.$$

We can take $\vec{p} = \sum_{i=1}^{k} a_i \vec{w}_i$ and $\vec{r} = \sum_{j=1}^{m} b_j \vec{v}_j$.

To prove uniqueness, suppose that we have $\vec{v} = \vec{p}_1 + \vec{r}_1$ with $\vec{p}_1 \in W$ and $\vec{r}_1 \in W^\perp$. Then

$$\vec{0} = \vec{v} - \vec{v} = (\vec{p} + \vec{r}) - (\vec{p}_1 + \vec{r}_1) = (\vec{p} - \vec{p}_1) + (\vec{r} - \vec{r}_1)$$

hence $\vec{p} - \vec{p}_1 = -(\vec{r} - \vec{r}_1)$. Now, $\vec{p} - \vec{p}_1 \in W$ and $-(\vec{r} - \vec{r}_1) \in W^\perp$ since W and $W^\perp$ are subspaces. So these identical vector are in $W \cap W^\perp$ hence they are both $\vec{0}$ by Proposition 4.6.2(b). Thus, $\vec{p} = \vec{p}_1$ and $\vec{r} = \vec{r}_1$.

If we apply Proposition 4.3.12 to $\vec{p}$, we find that $a_i = \dfrac{\langle \vec{v}, \vec{w}_i \rangle}{\|\vec{w}_i\|^2}$, completing the proof. □

With this result, we are now finally able to address the second question at the end of Section 4.4 and properly define the projection onto a subspace W of an inner product space V.

Definition 4.6.7

Projection onto a Subspace, proj_W**, Perpendicular Vector with Respect to a Subspace,** perp_W

Let V be an inner product space and let $W \subseteq V$ be a subspace. Let $\{\vec{w}_1, \ldots, \vec{w}_k\}$ be an orthogonal basis for W. Let $\vec{v} \in V$. The **projection of $\vec{v}$ onto W** and the **perpendicular vector of $\vec{v}$ with respect to W** are defined to be

$$\text{proj}_W(\vec{v}) = \text{proj}_{\vec{w}_1}(\vec{v}) + \cdots + \text{proj}_{\vec{w}_k}(\vec{v}) \quad \text{and} \quad \text{perp}_W(\vec{v}) = \vec{v} - \text{proj}_W(\vec{v}).$$

respectively.

Part (a) of the next result shows that $\text{proj}_W(\vec{v})$ (and hence $\text{perp}_W(\vec{v})$) is well-defined and does not depend on the choice of orthogonal basis for W. Parts (b) and (c) also provide us with the alternative characterization of $\text{proj}_W(\vec{v})$ as being the unique vector in W that is *closest* to $\vec{v}$.

Proposition 4.6.8

Let V be a finite-dimensional inner product space, W a subspace of V and $\vec{v} \in V$. Let $\mathcal{B} = \{\vec{w}_1, \ldots, \vec{w}_k\}$ be an orthogonal basis for W and let

$$\vec{p} = \text{proj}_{\vec{w}_1}(\vec{v}) + \cdots + \text{proj}_{\vec{w}_k}(\vec{v}).$$

(a) If $\mathcal{C} = \{\vec{u}_1, \ldots, \vec{u}_k\}$ is an orthogonal basis for W, then

$$\text{proj}_{\vec{u}_1}(\vec{v}) + \cdots + \text{proj}_{\vec{u}_k}(\vec{v}) = \vec{p}.$$

In particular, the definition of $\text{proj}_W(\vec{v})$ in Definition 4.6.7 is independent of the choice of orthogonal basis for W.

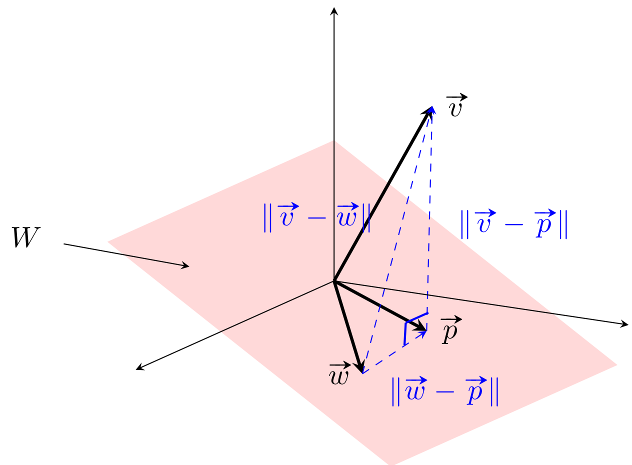
(b) For all $\vec{w} \in W$, $\|\vec{v} - \vec{p}\| \leq \|\vec{v} - \vec{w}\|$.

(c) If $\|\vec{v} - \vec{p}\| = \|\vec{v} - \vec{w}\|$ for some $\vec{w} \in W$, then $\vec{w} = \vec{p}$.

Proof: (a) The vector $\vec{p}$ in this proposition is exactly the same as the vector $\vec{p}$ in the Orthogonal Decomposition theorem. This result follows from the uniqueness of $\vec{p}$.

(b) Notice that $\vec{v} - \vec{w} = (\vec{v} - \vec{p}) + (\vec{p} - \vec{w})$. Now, $\vec{r} = \vec{v} - \vec{p}$ is in $W^{\perp}$ and $\vec{p} - \vec{w}$ is in W since W is a subspace. So $\vec{v} - \vec{p} \perp \vec{p} - \vec{w}$, and thus the Pythagorean theorem yields

$$\|\vec{v} - \vec{w}\|^2 = \|\vec{v} - \vec{p}\|^2 + \|\vec{p} - \vec{w}\|^2 \geq \|\vec{v} - \vec{p}\|^2 .$$



(c) The only way we have equality in (b) is if $\|\vec{p} - \vec{w}\|^2 = 0$, or equivalently, if $\vec{w} = \vec{p}$, as required.

□

It is important to keep in mind that our definition of $\text{proj}_W(\vec{v})$ requires an *orthogonal* basis for W.

EXERCISE

Let $\{\vec{z}_1, \ldots, \vec{z}_k\}$ be a basis for W that is *not* orthogonal. Show that there exists a vector $\vec{v} \in V$ such that $\text{proj}_{\vec{z}_1}(\vec{v}) + \cdots + \text{proj}_{\vec{z}_k}(\vec{v}) \neq \text{proj}_W(\vec{v})$.

Example 4.6.9

Consider the vector space $\mathcal{P}_3(\mathbb{R})$ with the inner product $\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)\,dx$. We know from an earlier example that $1 \perp x$, so $\{1, x\}$ is an orthogonal basis for $W = \text{Span}(\{1, x\})$. Let's find the projection of x^2 onto W.

We have

$$\text{proj}_W(x^2) = \frac{\langle x^2, 1 \rangle}{\|1\|^2} 1 + \frac{\langle x^2, x \rangle}{\|x\|^2} x.$$

It can be checked that $\langle x^2, 1 \rangle = \frac{2}{3}$, $\langle x^2, x \rangle = 0$, and $\|1\|^2 = 2$. Therefore

$$\text{proj}_W(x^2) = \frac{1}{2}\frac{2}{3}1 = \frac{1}{3}$$

and

$$\text{perp}_W(x^2) = x^2 - \frac{1}{3}.$$

This tells us that the closest vector in W to x^2 is the vector $\frac{1}{3}$. Go figure!

We close this section with a neat example of projection.

Example 4.6.10 **(Fourier expansion)**

Consider the vector space $V = \mathcal{C}([-\pi, \pi])$ of continuous function $f\colon [-\pi, \pi] \to \mathbb{R}$ equipped with the inner product $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx$. You can check that $S = \{1, \sin x, \cos x\}$ is an orthogonal subset of V.

Let $W = \text{Span}(S)$. Let's find the projection of the function $f(x) = x$ onto W. We have

$$\text{proj}_W(f) = \text{proj}_1(x) + \text{proj}_{\sin x}(x) + \text{proj}_{\cos x}(x).$$

We leave it to you to show that

$$\text{proj}_1(x) = 0, \quad \text{proj}_{\sin x}(x) = 2 \quad \text{and} \quad \text{proj}_{\cos x}(x) = 0.$$

Hence $\text{proj}_W(x) = 2\sin x$. This is the beginning of the so-called *Fourier expansion* of $f(x) = x$ on the interval $[-\pi, \pi]$, which is a way of approximating f with sinusoidal functions.

We can project onto the subspace spanned by the orthogonal set

$$\{1,\ \cos(x),\ \sin(x),\ \cos(2x),\ \sin(2x),\ \ldots,\ \cos(nx),\ \sin(nx)\}$$

to obtain more terms of the Fourier expansion of f. This is an entry point to a very rich area of mathematics that has broad applications to physics, computer science, engineering, finance, and so on. Matters become much more interesting once we allow ourselves to dabble in infinite-dimensional vector spaces, which will grant us access to infinite Fourier expansions.

EXERCISE

Let V be the inner product space from the previous example.

(a) Show that, for all $n \geq 1$, the set

$$S_n = \{1,\ \cos(x),\ \sin(x),\ \cos(2x),\ \sin(2x),\ \ldots,\ \cos(nx),\ \sin(nx)\}$$

is an orthogonal set in V.

(b) Let $W_n = \text{Span}(S_n)$, where S_n is as in part (a). Find $\text{proj}_{W_n}(f)$, where $f \in V$ is the function $f(x) = |x|$.

(c) On the same set of axes, plot the graphs of $f(x) = |x|$ and $\text{proj}_{W_n}(f)$ for $n = 1, 3, 9$. What do you notice?

4.7 Application: Method of Least Squares

In this section we will work over $\mathbb{F} = \mathbb{R}$.

Suppose we have a system of equations expressed in matrix form as $A\vec{x} = \vec{b}$. We know that this system has a solution if and only if $\vec{b} \in \text{Col}(A)$. If $\vec{b} \notin \text{Col}(A)$, then there is a way to obtain an *approximate* solution to this system. We first define $\vec{p} = \text{proj}_{\text{Col}(A)}(\vec{b})$ to be the vector in $\text{Col}(A)$ closest to $\vec{b}$. Then we know that the system $A\vec{x} = \vec{p}$ has a solution.

Definition 4.7.1 **Least Squares Solution**

Let $A \in M_{m\times n}(\mathbb{R})$ and $\vec{b} \in \mathbb{R}^m$. The vector $\vec{s} \in \mathbb{R}^n$ is called a **least squares solution** to $A\vec{x} = \vec{b}$ if it is a solution to the system $A\vec{x} = \vec{p}$, where $\vec{p} = \text{proj}_{\text{Col}(A)}(\vec{b})$.

We think of least squares solutions as being approximate solutions to the system $A\vec{x} = \vec{b}$. The reason for the name stems from the fact that a least squares solution $\vec{x} = \vec{s}$ minimizes the quantity $\|A\vec{x} - \vec{b}\|^2$, which is a sum of squares. Indeed, since $A\vec{x}$ represents an arbitrary element of $\text{Col}(A)$, the distance between $A\vec{x}$ and $\vec{b}$ is minimized precisely when $A\vec{x} = \text{proj}_{\text{Col}(A)}(\vec{b})$, by Proposition 4.6.8(b) and (c).

Example 4.7.2

Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ and observe that $\text{Col}(A) = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$. Let $\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then $\vec{b} \notin \text{Col}(A)$ so the equation $A\vec{x} = \vec{b}$ does not have a solution.

Let's find $\vec{p} = \text{proj}_{\text{Col}(A)}(\vec{b})$. This is simply equal to

$$\vec{p} = \text{proj}_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \frac{\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Now the equation $A\vec{x} = \vec{p}$ has a solution—in fact, it has infinitely many. The solution set is given by

$$S = \left\{\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 1 \end{bmatrix} : t \in \mathbb{R}\right\}$$

as you can check. Any $\vec{s} \in S$ is a least squares solution to the system $A\vec{x} = \vec{b}$.

Let's give an alternative, and more practical, characterization of least squares solutions.

Proposition 4.7.3

Let $A \in M_{m\times n}(\mathbb{R})$ and $\vec{b} \in \mathbb{R}^m$. The vector $\vec{s} \in \mathbb{R}^n$ is a least squares solution to $A\vec{x} = \vec{b}$ if and only if it is a solution to $A^TA\vec{x} = A^T\vec{b}$.

Proof: Suppose that $\vec{s}$ is a least squares solution to $A\vec{x} = \vec{b}$. Then $A\vec{s} = \vec{p}$ where $\vec{p} = \text{proj}_{\text{Col}(A)}(\vec{b})$. Now, $\vec{b} - \vec{p} = \text{perp}_{\text{Col}(A)}(\vec{b})$ is in $(\text{Col}(A))^\perp$ hence is orthogonal to the columns of A. So, letting $\vec{a}_i$ denote the i^{th} column of A, we have

$$\vec{a}_i \cdot (\vec{b} - \vec{p}) = 0 \iff \vec{a}_i^T(\vec{b} - \vec{p}) = 0$$

for all i. That is, $A^T(\vec{b}-\vec{p})=\vec{0}$, or equivalently $A^T(\vec{b}-A\vec{s})=\vec{0}$. Hence $A^T\vec{b}=A^TA\vec{s}$, as required.

Conversely, if $A^TA\vec{s}=A^T\vec{b}$, then $A^T(A\vec{s}-\vec{b})=\vec{0}$ and by the same reasoning as above, we see that $A\vec{s}-\vec{b}$ is orthogonal to the columns of A and hence must be in $\mathrm{Col}(A)^\perp$ by Proposition 4.6.5(a). Since $A\vec{s}$ is in $\mathrm{Col}(A)$, it follows that $A\vec{s}$ must be the projection of $\vec{b}$ onto $\mathrm{Col}(A)$ (see exercise below). That is, $A\vec{s}=\vec{p}$, and so $\vec{s}$ is a least squares solution. This completes the proof. □

We used the following observation to complete the proof of the preceding proposition.

EXERCISE

Let V be an inner product space and W a subspace. Let $\vec{v}\in V$ and $\vec{w}\in W$. Prove that $\vec{w}=\mathrm{proj}_W(\vec{v})$ if and only if $\vec{v}-\vec{w}\in W^\perp$.

Example 4.7.4 Let $A=\begin{bmatrix}1&2\\1&2\end{bmatrix}$ and $\vec{b}=\begin{bmatrix}1\\0\end{bmatrix}$, as in the previous example. Then

$$A^TA=\begin{bmatrix}2&4\\4&8\end{bmatrix}\quad\text{and}\quad A^T\vec{b}=\begin{bmatrix}1\\2\end{bmatrix}.$$

The system

$$\begin{bmatrix}2&4\\4&8\end{bmatrix}\vec{x}=\begin{bmatrix}1\\2\end{bmatrix}$$

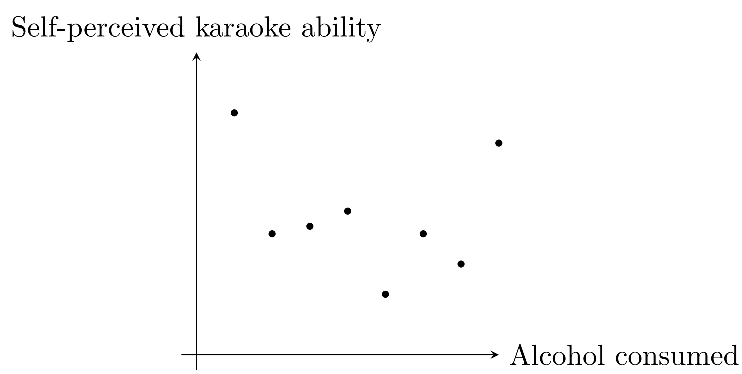
has solution set

$$S=\left\{\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}+t\begin{bmatrix}-2\\1\end{bmatrix} : t\in\mathbb{R}\right\}$$

which coincides with what we found in Example 4.7.2 above, confirming Proposition 4.7.3.

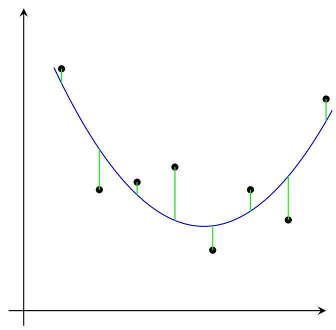
4.7.1 Least Squares Curve Fitting

Suppose we're doing a super-serious study, and we've gathered a collection of data which is looking for some kind of relationship between "self-perceived karaoke ability" and "alcohol consumed." The data we've collected looks like this when plotted:



Our goal is to model this data by some quadratic equation $y = a + bx + cx^2$ where y is the perceived karaoke ability and x is the alcohol consumed. After all, we would expect this to occur in reality: a person while sober thinks they're quite good, after a couple of drinks is aware they will be slurring a little, but after drinking more will begin to think they are god's gift to vocal performance!

So, we would like to find a quadratic that looks something like the blue curve:



Furthermore, we would like such a quadratic to make the lengths of the vertical green lines as small as possible, since the vertical green lines represent the error between our model and the experimental data.

So let's say we had the data points $(x_1, y_1), \dots, (x_n, y_n)$ which we want to approximate by $y = a + bx + cx^2$. We want this curve to best fit the points. But what do we mean by "best fit"? We want to choose the curve that minimizes the sum of the squares of the lengths of the vertical green lines:

$$(y_1 - (a + bx_1 + cx_1^2))^2 + \cdots + (y_n - (a + bx_n + cx_n^2))^2.$$

(Hence the name: *least squares*!) This looks an awful lot like a norm in $\mathbb{R}^n$ with respect to the dot product. In fact, if we let

$$\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \quad \vec{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \vec{x}^2 = \begin{bmatrix} x_1^2 \\ \vdots \\ x_n^2 \end{bmatrix}$$

be vectors in $\mathbb{R}^n$, then minimizing the sum of the squares of the errors (the vertical green bars) is the same as minimizing

$$\left\| \vec{y} - (a\vec{1} + b\vec{x} + c\vec{x}^2) \right\|^2$$

with respect to the dot product. In other words, to find a, b, and c, we need to find the vector on the subspace $W = \operatorname{Span}(\{\vec{1}, \vec{x}, \vec{x}^2\})$ closest to the vector $\vec{y}$. We know how to do this! Putting all of these observations together, we find a, b, and c by setting

$$a\vec{1} + b\vec{x} + c\vec{x}^2 = \operatorname{proj}_W(\vec{y}).$$

If we let

$$X = \begin{bmatrix} \vec{1} & \vec{x} & \vec{x}^2 \end{bmatrix} \quad \text{and} \quad \vec{s} = \begin{bmatrix} a \\ b \\ c \end{bmatrix},$$

then the above equation becomes

$$X\vec{s} = \text{proj}_W(\vec{y}).$$

That is, we are looking for a least squares solution $\vec{s}$ to the equation $X\vec{x} = \vec{y}$! By Proposition 4.7.3, we know that this is equivalent to finding a solution to

$$X^T X\vec{s} = X^T\vec{y}.$$

Now, *if* the 3×3 matrix $X^T X$ were invertible, then we'd be able to get our solution $\vec{s}$ as

$$\vec{s} = (X^T X)^{-1} X^T \vec{y}.$$

Let's see this in action.

Example 4.7.5 Suppose we have the following data:

x	-1	0	1	2
y	4	1	1	-1

Let's first try to approximate this data set by a linear equation $y = a + bx$. So we let

$$\vec{s} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad X = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad \text{and} \quad \vec{y} = \begin{bmatrix} 4 \\ 1 \\ 1 \\ -1 \end{bmatrix}.$$

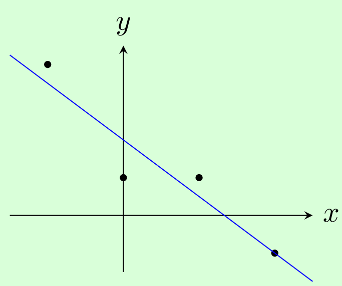
Then

$$X^T X = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix},$$

which is an invertible matrix! Thus,

$$\vec{s} = (X^T X)^{-1} X^T \vec{y} = \left(\begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix} \right)^{-1} \begin{bmatrix} 5 \\ -5 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ -5 \end{bmatrix} = \begin{bmatrix} 2 \\ -\frac{3}{2} \end{bmatrix}.$$

Therefore $y = 2 - \frac{3}{2}x$ is the line of best fit to the given data. Let's see what this line looks like.



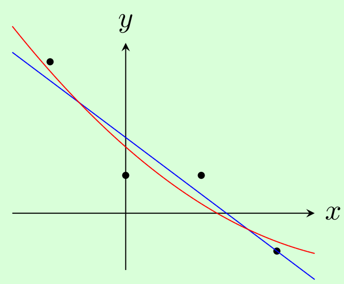
While this is good, maybe it's not as good as we'd like! Let's see if we can do better approximating the data by a quadratic equation $y = a + bx + cx^2$. This time we have

$$\vec{s} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad X = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}, \quad \text{and} \quad \vec{y} = \begin{bmatrix} 4 \\ 1 \\ 1 \\ -1 \end{bmatrix}.$$

Again, $X^T X$ turns out to be invertible (we'll let you check this), and so we get

$$\vec{s} = (X^T X)^{-1} X^T \vec{y} = \begin{bmatrix} \frac{7}{4} \\ -\frac{7}{4} \\ \frac{1}{4} \end{bmatrix}.$$

Therefore the quadratic curve of best fit is $y = \frac{7}{4} - \frac{7}{4}x + \frac{1}{4}x^2$. Here's a plot:



That's a little better!

In general, suppose we have some data points

x	x_1	$\cdots$	x_n
y	y_1	$\cdots$	y_n

and we want to find the equation $y = a_0 + a_1 x + \cdots + a_k x^k$ of best fit to this data. Let

$$\vec{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \vec{x}^2 = \begin{bmatrix} x_1^2 \\ \vdots \\ x_n^2 \end{bmatrix}, \ldots, \vec{x}^k = \begin{bmatrix} x_1^k \\ \vdots \\ x_n^k \end{bmatrix}, \vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \quad \text{and} \quad \vec{s} = \begin{bmatrix} a_0 \\ \vdots \\ a_k \end{bmatrix}.$$

Let $X = \begin{bmatrix} \vec{1} & \vec{x} & \cdots & \vec{x}^k \end{bmatrix}$. If $X^T X$ is invertible, then $\vec{s} = (X^T X)^{-1} X^T \vec{y}$ gives the coefficients of the equation of best fit.

A natural question, then, is: When is the matrix $X^T X$ invertible?

Proposition 4.7.6 Let $X \in M_{m\times n}(\mathbb{R})$. Then $X^T X \in M_{n\times n}(\mathbb{R})$ is invertible if and only if the columns of X are linearly independent.

Proof: The matrix $X^T X$ will be invertible if and only if the columns of $X^T X$ are linearly independent, which will be the case if and only if $\text{Null}(X^T X) = \{\vec{0}\}$. We will show that

$\text{Null}(X^T X) = \text{Null}(X)$. This will show that $X^T X$ is invertible if and only if the columns of X are linearly independent, as desired.

If $\vec{x} \in \text{Null}(X)$, then $X^T X\vec{x} = X^T\vec{0} = \vec{0}$, so $\vec{x} \in \text{Null}(X^T X)$ and $\text{Null}(X) \subseteq \text{Null}(X^T X)$. Conversely, if $\vec{x} \in \text{Null}(X^T X)$ then $\vec{x}^T X^T X\vec{x} = \vec{x}^T\vec{0} = 0$ hence $(X\vec{x})^T(X\vec{x}) = 0$, or equivalently, $(X\vec{x})\cdot(X\vec{x}) = 0$. Thus $X\vec{x} = \vec{0}$, so $\vec{x} \in \text{Null}(X)$ and $\text{Null}(X^T X) \subseteq \text{Null}(X)$, which completes the proof. $\square$

The next exercise addresses the issue in the case of the best quadratic fit.

EXERCISE

Let $X = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 \end{bmatrix} \in M_{n\times 3}(\mathbb{R})$.

(a) Assume that $n \geq 3$. Show that $X^T X$ is invertible if and only if at least three of the x_i are distinct.

(b) Show that $X^T X$ is never invertible if $n < 3$.

The assertions in this exercise should seem plausible. For instance, part (b) says that there is no *unique* best fitting quadratic curve (parabola) through two or fewer points.

Chapter 5

Orthogonal Diagonalization

5.1 Orthogonal and Unitary Matrices

When working with a linear operator $L\colon V \to V$ on a finite-dimensional vector space, we have seen in Chapter 3 how convenient it can be to have a basis $\mathcal{D}$ for V consisting of eigenvectors of L. If V also happens to be an inner product space, then we've learned in Chapter 4 how useful it can be to have an orthonormal basis $\mathcal{B}$ for V. For a typical operator, its eigenvectors will not be orthogonal, and so we will have to choose to either work with eigenvectors or with orthogonal vectors.

So a natural question, then, is: When does a linear operator $L\colon V \to V$ on a finite-dimensional inner product space over $\mathbb{F}$ admit a basis for V consisting of orthogonal eigenvectors? If $\mathbb{F} = \mathbb{R}$, the answer is: if and only if L is *self-adjoint*. If $\mathbb{F} = \mathbb{C}$, the answer is: if and only if L is *normal*. We'll explain what this means in due course.

For now let's take up the question of when a *matrix* $A \in M_{n\times n}(\mathbb{F})$ admits an orthonormal basis of eigenvectors for $\mathbb{F}^n$, where $\mathbb{F}^n$ is endowed with the standard inner product. This will *almost* fully solve the first problem, except for one issue about working with general inner products—which we're going to postpone to the end of the chapter.

Now recall that $A \in M_{n\times n}(\mathbb{F})$ admits a basis of eigenvectors for $\mathbb{F}^n$ precisely when A is diagonalizable (Theorem 3.2.7), that is, if and only if there is an invertible matrix P such that $P^{-1}AP$ is diagonal. When this is the case, the columns of P will form a basis of $\mathbb{F}^n$ consisting of eigenvectors of A. Thus being able to find a basis of eigenvectors for $\mathbb{F}^n$ is effectively equivalent to being able to construct this matrix P. Supposing we can find an *orthonormal* basis of eigenvectors, what can we then say about P? First, a quick definition.

Definition 5.1.1
Adjoint, A^*, $A^\dagger$

If $A \in M_{n\times n}(\mathbb{F})$, the **adjoint** of A is the matrix $\overline{A^T} \in M_{n\times n}(\mathbb{F})$. It is denoted by A^* (read *A star*). (In some texts the notation $A^\dagger$, read *A dagger*, is also used.)

The bar denotes complex conjugation. So $A^* = \overline{A^T}$ is the matrix whose entries are the complex conjugates of the entries of A^T. For example,

$$\begin{bmatrix} 2 & 4 \\ -i & 2+i \end{bmatrix}^* = \begin{bmatrix} 2 & i \\ 4 & 2-i \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 1-i \\ 1+i & 3 \end{bmatrix}^* = \begin{bmatrix} 2 & 1-i \\ 1+i & 3 \end{bmatrix}.$$

If $\mathbb{F} = \mathbb{R}$ then $A^* = A^T$ is just the transpose of A.

Proposition 5.1.2

Let $P \in M_{n\times n}(\mathbb{F})$. Equip $\mathbb{F}^n$ with the standard inner product. Then the following properties are equivalent:

(a) The columns of P form an orthonormal basis for $\mathbb{F}^n$.

(b) $P^* = P^{-1}$.

(c) The rows of P form an orthonormal basis for $\mathbb{F}^n$.

Proof: The key observation is that if $\vec{v}, \vec{w} \in \mathbb{F}^n$ then their standard inner product may be expressed by

$$\langle \vec{v}, \vec{w} \rangle = \overline{\vec{w}^T}\vec{v},$$

where we are interpreting the right-side as the product of an $n \times 1$ and a $1 \times n$ matrix. (Technically, this results in a 1×1 matrix, but we will tacitly identify this with a number in $\mathbb{F}$!)

We will prove the equivalence of (a) and (b). Let $P = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$. Then

$$P^*P = \begin{bmatrix} \overline{\vec{v}_1^T} \\ \vdots \\ \overline{\vec{v}_n^T} \end{bmatrix} \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix} = \begin{bmatrix} \langle \vec{v}_1, \vec{v}_1 \rangle & \cdots & \langle \vec{v}_n, \vec{v}_1 \rangle \\ \vdots & \ddots & \vdots \\ \langle \vec{v}_1, \vec{v}_n \rangle & \cdots & \langle \vec{v}_n, \vec{v}_n \rangle \end{bmatrix}.$$

From this we see that $P^*P = I_n$ if and only if

$$\langle \vec{v}_i, \vec{v}_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

equivalently, if and only if $\{\vec{v}_1, \dots, \vec{v}_n\}$ is an orthonormal set, which is precisely what we wanted to prove.

The proof of the equivalence of (c) with (a) and (b) is left as an exercise. $\square$

EXERCISE

Show that properties (b) and (c) of Proposition 5.1.2 are equivalent. (Hint: The rows of P are the columns of P^T.) Observe that since (a) and (b) are equivalent, this also proves the equivalence of (c) and (a).

Let's give a name to matrices that satisfy any of the equivalent properties in Proposition 5.1.2. It's customary to separate the case where $\mathbb{F} = \mathbb{R}$.

Definition 5.1.3

Unitary Matrix, Orthogonal Matrix

A matrix $U \in M_{n\times n}(\mathbb{F})$ is called a **unitary matrix** if $U^* = U^{-1}$.

A matrix $Q \in M_{n\times n}(\mathbb{R})$ is called an **orthogonal matrix** if $Q^T = Q^{-1}$.

Of course, an orthogonal matrix is by definition also a unitary matrix. However, we generally (but not always) reserve the adjective "unitary" for when we are working with complex matrices. You might argue that an orthogonal matrix should be called an *orthonormal* matrix, since its columns are in fact orthonormal and not just orthogonal. You would have a point. Alas, the definition "orthogonal" is deeply entrenched in the literature.

Example 5.1.4 The $n \times n$ identity matrix is unitary. (In fact, orthogonal.)

Example 5.1.5 Let $U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}}i & \frac{1}{\sqrt{2}}i \end{bmatrix}$. Then U is unitary since

$$UU^* = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which proves that $U^* = U^{-1}$.

Example 5.1.6 Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Notice that the columns of A are orthogonal. However, A is **not** an orthogonal matrix since its columns are not orthonormal!

Example 5.1.7 For $\theta \in \mathbb{R}$, let $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ (the rotation-by-θ-counterclockwise matrix in $\mathbb{R}^2$). Then R_θ is orthogonal. We leave it to you to check that

$$R_\theta R_\theta^T = I.$$

Unitary matrices turn out to be interesting for a variety of reasons. When viewed as linear maps with respect to the standard basis in $\mathbb{F}^n$, they do not affect the standard inner product. That is, unitary matrices preserve length and angle!

Proposition 5.1.8 Let $U \in M_{n\times n}(\mathbb{F})$ be a unitary matrix and consider $\mathbb{F}^n$ with the standard inner product. Then:

(a) $\langle U\vec{v}, U\vec{u}\rangle = \langle \vec{v}, \vec{u}\rangle$ for all $\vec{v}, \vec{u} \in \mathbb{F}^n$.

(b) $\|U\vec{v}\| = \|\vec{v}\|$ for all $\vec{v} \in \mathbb{F}^n$.

Proof: (a) Since the standard inner product on $\mathbb{F}^n$ is given by $\langle \vec{v}, \vec{u}\rangle = \overline{\vec{u}^T}\vec{v}$, we have

$$\langle U\vec{v}, U\vec{u}\rangle = \overline{(U\vec{u})^T}U\vec{v} = \overline{\vec{u}^T}U^*U\vec{v} = \overline{\vec{u}^T}\vec{v} = \langle \vec{v}, \vec{u}\rangle$$

completing the proof.

(b) Using part (a), we have

$$\|U\vec{v}\| = \sqrt{\langle U\vec{v}, U\vec{v}\rangle} = \sqrt{\langle \vec{v}, \vec{v}\rangle} = \|\vec{v}\|.$$

□

EXERCISE

Let $Q \in M_{n\times n}(\mathbb{R})$ be an orthogonal matrix and let $\vec{v}, \vec{u} \in \mathbb{R}^n$ be non-zero vectors. Prove that the angle between $\vec{v}$ and $\vec{u}$ is equal to the angle between $Q\vec{v}$ and $Q\vec{u}$.

Let's close this introductory section with some properties of adjoints.

Proposition 5.1.9 **(Properties of the Adjoint)**

Let $A, B \in M_{n\times n}(\mathbb{F})$. Then:

(a) $(A+B)^* = A^* + B^*$.

(b) $(AB)^* = B^*A^*$.

(c) $(A^*)^* = A$.

(d) $(\alpha A)^* = \overline{\alpha}A^*$ for all $\alpha \in \mathbb{F}$.

EXERCISE

Prove Proposition 5.1.9.

We've singled out this next property because, although easy to prove, it's more subtle.

Proposition 5.1.10 **(The Fundamental Property of the Adjoint of a Matrix)**

Let $A \in M_{n\times n}(\mathbb{F})$. Equip $\mathbb{F}^n$ with the standard inner product. Then for all $\vec{v}, \vec{w} \in \mathbb{F}^n$,

$$\langle A\vec{v}, \vec{w}\rangle = \langle \vec{v}, A^*\vec{w}\rangle .$$

Proof: This follows from observing that the standard inner product on $\mathbb{F}^n$ is given by

$$\langle \vec{v}, \vec{w}\rangle = \overline{\vec{w}^T}\vec{v} = \vec{w}^*\vec{v}$$

(a fact we have already used). So,

$$\langle A\vec{v}, \vec{w}\rangle = \vec{w}^*A\vec{v} = (\vec{w}^*(A^*)^*)\vec{v} = (A^*\vec{w})^*\vec{v} = \langle \vec{v}, A^*\vec{w}\rangle .$$

□

5.2 Schur's Triangularization Theorem

We have just seen that a unitary matrix can be viewed as a special kind of change of basis matrix, one that preserves length and angles. While it is not true that every matrix is diagonalizable, we will now see that every matrix is upper-triangularizable (at least over $\mathbb{C}$), that is, for every matrix we can find a basis for $\mathbb{C}^n$ with respect to which the matrix is upper-triangular. Even better, we can choose this basis to be orthonormal (with respect to the standard inner product)—meaning, the change of basis matrix will be unitary. We can sometimes do this over $\mathbb{R}$, too.

Theorem 5.2.1 **(Schur's Triangularization Theorem)**

Let $A \in M_{n\times n}(\mathbb{F})$. There is a unitary matrix $U \in M_{n\times n}(\mathbb{C})$ and an upper-triangular matrix $T \in M_{n\times n}(\mathbb{C})$ such that $U^*AU = T$:

$$U^*AU = T = \begin{bmatrix} \lambda_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & \lambda_n \end{bmatrix}.$$

The diagonal entries λ_i of T are the complex eigenvalues of A (repeated according to multiplicity).

If $A \in M_{n\times n}(\mathbb{R})$, and if all the complex eigenvalues λ_i of A are in $\mathbb{R}$, then U may be chosen to be real and orthogonal and T will be in $M_{n\times n}(\mathbb{R})$.

NOTE: The $*$ entries in the matrix just indicate that any element of $\mathbb{F}$ can appear in those entries of the matrix.

REMARK

In stating this theorem, we're using the fact that the characteristic polynomial of a matrix $A \in M_{n\times n}(\mathbb{F})$ will be a degree n polynomial with complex coefficients. Hence, by the fundamental theorem of algebra, A will have n eigenvalues in $\mathbb{C}$ (possibly repeated).

This applies in particular to the case where $A \in M_{n\times n}(\mathbb{R})$. Such a matrix will typically have non-real roots. Schur's theorem shows that A can be triangularized using a unitary matrix $U \in M_{n\times n}(\mathbb{C})$. In the special case where the n eigenvalues of A are all real, the theorem asserts that we can arrange for U to be in $M_{n\times n}(\mathbb{R})$.

Proof of Theorem 5.2.1: We will proceed by induction on n. For $n = 1$, this is clearly true since every 1×1 matrix is upper-triangular.

Now suppose A is an $n \times n$ matrix, and assume that the theorem is true for all $(n-1) \times (n-1)$ matrices. Let $\vec{v}_1$ be a unit eigenvector of A with eigenvalue λ. Extend $\{\vec{v}_1\}$ to a basis for $\mathbb{C}^n$ and perform the Gram–Schmidt procedure to obtain an orthonormal basis $\{\vec{v}_1, \vec{w}_2, \dots, \vec{w}_n\}$.

Let $V_1 = \begin{bmatrix} \vec{v}_1 & \vec{w}_2 & \cdots & \vec{w}_n \end{bmatrix}$. Since the columns are orthonormal, V_1 is a unitary matrix. We then have

$$\begin{aligned}
V_1^* A V_1 &= \begin{bmatrix} \vec{v}_1{}^* \\ \vec{w}_2{}^* \\ \vdots \\ \vec{w}_n{}^* \end{bmatrix} A \begin{bmatrix} \vec{v}_1 & \vec{w}_2 & \cdots & \vec{w}_n \end{bmatrix} \\
&= \begin{bmatrix} \vec{v}_1{}^* \\ \vec{w}_2{}^* \\ \vdots \\ \vec{w}_n{}^* \end{bmatrix} \begin{bmatrix} A\vec{v}_1 & A\vec{w}_2 & \cdots & A\vec{w}_n \end{bmatrix} \\
&= \begin{bmatrix} \vec{v}_1{}^* \\ \vec{w}_2{}^* \\ \vdots \\ \vec{w}_n{}^* \end{bmatrix} \begin{bmatrix} \lambda\vec{v}_1 & A\vec{w}_2 & \cdots & A\vec{w}_n \end{bmatrix} \\
&= \begin{bmatrix} \langle \lambda\vec{v}_1, \vec{v}_1\rangle & \langle A\vec{w}_2, \vec{v}_1\rangle & \cdots & \langle A\vec{w}_n, \vec{v}_1\rangle \\ \langle \lambda\vec{v}_1, \vec{w}_2\rangle & \langle A\vec{w}_2, \vec{w}_2\rangle & \cdots & \langle A\vec{w}_n, \vec{w}_2\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \lambda\vec{v}_1, \vec{w}_n\rangle & \langle A\vec{w}_2, \vec{w}_n\rangle & \cdots & \langle A\vec{w}_n, \vec{w}_n\rangle \end{bmatrix} \\
&= \left[\begin{array}{c|ccc} \lambda & * & * & * \\ \hline 0 & & & \\ \vdots & & B & \\ 0 & & & \end{array}\right].
\end{aligned}$$

Now B is an $(n-1) \times (n-1)$ matrix, so by the inductive hypothesis there is a unitary matrix V_2 such that $V_2^* B V_2 = T_2$ where T_2 is upper-triangular. Let

$$U = V_1 \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & V_2 \end{array}\right].$$

Then $U^* U = I_n$ (why?) so U is unitary, and we have

$$\begin{aligned}
U^* A U &= \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & V_2 \end{array}\right]^* V_1^* A V_1 \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & V_2 \end{array}\right] \\
&= \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & V_2^* \end{array}\right] \left[\begin{array}{c|c} \lambda & * \\ \hline 0 & B \end{array}\right] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & V_2 \end{array}\right] \\
&= \left[\begin{array}{c|c} \lambda & * \\ \hline 0 & V_2^* B V_2 \end{array}\right] \\
&= \left[\begin{array}{c|c} \lambda & * \\ \hline 0 & T_2 \end{array}\right] = T.
\end{aligned}$$

Since this last matrix T is upper-triangular, the first part of the theorem has been proved by the principle of mathematical induction.

Now observe that since U is unitary, $U^* = U^{-1}$ so we see that A is similar to T. Consequently, the eigenvalues of A and T are the same. However, since T is upper-triangular, its eigenvalues are its diagonal entries. This proves the second part of the theorem.

Finally, if A is real with real eigenvalues, then in the inductive step above, $\vec{v}_1$ will be in $\mathbb{R}^n$, λ will be in $\mathbb{R}$, and all the consequent steps can be carried out in $\mathbb{R}^n$ with real arithmetic. We'll leave the careful verification to you. $\square$

The take home message is that every matrix is similar (over $\mathbb{C}$!) to an upper-triangular one. (But not to a unique upper-triangular matrix. See the next example.)

Example 5.2.2 Let $T_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ and $T_2 = \begin{bmatrix} 2 & -1 & \sqrt{2} \\ 0 & 1 & -\sqrt{2} \\ 0 & 0 & 2 \end{bmatrix}$. Then you can check that $U^*T_1U = T_2$ with the unitary matrix

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So unlike with diagonalization, where the resulting diagonal matrix is unique up to a re-ordering of diagonal entries, the situation with triangularization is a bit more complicated.

Example 5.2.3 To illustrate the power of Schur's theorem, let's give a quick proof of the fact that, for any matrix $A \in M_{n\times n}(\mathbb{F})$, its determinant is the product of its complex eigenvalues and its trace is the sum of its complex eigenvalues. (This was given without proof in Corollary 3.1.16.)

By Schur's theorem we know there is a unitary matrix U and an upper-triangular matrix

$$T = \begin{bmatrix} \lambda_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & \lambda_n \end{bmatrix}$$

such that $U^*AU = T$. The characteristic polynomial of T is

$$C_T(\lambda) = (\lambda_1 - \lambda)\cdots(\lambda_n - \lambda)$$

so the λ_i are the eigenvalues of T. Furthermore we have

$$\mathrm{tr}(T) = \lambda_1 + \cdots + \lambda_n \quad \text{and} \quad \det(T) = \lambda_1 \cdots \lambda_n.$$

Since A and T are similar, they have the same eigenvalues, determinant and trace, completing the proof.

At this point we would be remiss not to mention one of the more intriguing consequences of Schur's theorem: the Cayley–Hamilton theorem, which says that if you plug a matrix into its own characteristic polynomial you get the zero matrix. That is, A is a "root" of its own characteristic polynomial!

For example, if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then its characteristic polynomial is $C_A(\lambda) = \lambda^2 - 5\lambda - 2$. If we plug $\lambda = A$ into the expression $\lambda^2 - 5\lambda - 2I$, we get:

$$A^2 - 5A - 2I = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

This is a general phenomenon. Let's agree that if $p(x) = a_0 + a_1x + \cdots + a_nx^n$ is a polynomial and if A is a square matrix, then $p(A) = a_0I + a_1A + \cdots + a_nA^n$.

Theorem 5.2.4 **(Cayley–Hamilton Theorem)**

Let $A \in M_{n\times n}(\mathbb{F})$. Then $C_A(A) = 0_{n\times n}$.

Proof: By Schur's theorem, we can find a unitary matrix U and an upper triangular matrix

$$T = \begin{bmatrix} \lambda_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & \lambda_n \end{bmatrix}$$

in $M_{n\times n}(\mathbb{C})$ such that $A = UTU^*$. The diagonal entries of T are the complex eigenvalues of A, and therefore

$$C_A(\lambda) = (\lambda_1 - \lambda)\cdots(\lambda_n - \lambda).$$

As a result,

$$\begin{aligned} C_A(A) &= (\lambda_1 I - A)\cdots(\lambda_n I - A) \\ &= (\lambda_1 I - UTU^*)\cdots(\lambda_n I - UTU^*) \\ &= (\lambda_1 UU^* - UTU^*)\cdots(\lambda_n UU^* - UTU^*) \\ &= U(\lambda_1 I - T)U^* \cdots U(\lambda_n I - T)U^* \\ &= UC_A(T)U^*. \end{aligned}$$

So the proof will be complete if we can show that $C_A(T) = (\lambda_1 I - T)\cdots(\lambda_n I - T)$ is the zero matrix. Notice that the (i,i) entry of $T - \lambda_i I$ is zero. In particular, the first column of $\lambda_1 I - T$ is the zero column, and the first two columns of $\lambda_2 I - T$ are all zero except possibly for the $(1,2)$ entry. But then the upper left 2×2 block of $(\lambda_1 I - T)(\lambda_2 I - T)$ will be zero. Continuing inductively, we find that the upper left $k\times k$ block of $(\lambda_1 I - T)\cdots(\lambda_k I - T)$ is zero. Thus $C_A(T)$ is the $n\times n$ zero matrix, completing the proof. □

EXERCISE

What is wrong with the following "proof" of the Cayley–Hamilton theorem?

"Since $C_A(\lambda) = \det(A - \lambda I)$, if we let $\lambda = A$ we get $C_A(A) = \det(A - AI) = \det(0) = 0$."

We close this section by giving two fun applications of the Cayley–Hamilton theorem. The first one shows that we can express the inverse of an invertible matrix A as a linear combination of powers A; the second one illustrates a new approach to computing powers A^k of an arbitrary $n\times n$ matrix A. Recall that we had seen one approach to computing powers of diagonalizable matrices in Section 3.3. Now we have something that works for *any* matrix, diagonalizable or not.

Example 5.2.5 **(A^{-1} via Cayley–Hamilton)**

Let $A \in M_{n\times n}(\mathbb{F})$. If the characteristic polynomial of A is expressed as

$$C_A(\lambda) = c_0 + c_1\lambda + \cdots + c_n\lambda^n$$

then the Cayley–Hamilton theorem tells us that

$$c_0 I + c_1 A + \cdots + c_n A^n = 0.$$

If A is invertible, then $c_0 \neq 0$ (since 0 is not an eigenvalue of A and since the constant term of C_A is the product of the eigenvalues of A). Then we can multiply both sides of the above equation by $\frac{1}{c_0}A^{-1}$ to get

$$A^{-1} + \frac{c_1}{c_0} I + \frac{c_2}{c_0} A + \cdots + \frac{c_n}{c_0} A^{n-1} = 0.$$

Thus,

$$A^{-1} = -\frac{c_1}{c_0} I - \frac{c_2}{c_0} A - \cdots - \frac{c_n}{c_0} A^{n-1}.$$

For instance, if $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 2 & 1 & 1 \end{bmatrix}$ then the characteristic polynomial of A is

$$C_A(\lambda) = -\lambda^3 + 3\lambda^2 - 2\lambda - 4.$$

So

$$-A^3 + 3A^2 - 2A - 4I = 0.$$

Therefore, after multiplying by through by A^{-1} and re-arranging, we get

$$\begin{aligned} A^{-1} &= -\frac{1}{4}(A^2 - 3A + 2I) \\ &= -\frac{1}{4}\left(\begin{bmatrix} 3 & 5 & 0 \\ -2 & 0 & -2 \\ 4 & 6 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 6 & 3 \\ 0 & 3 & -3 \\ 6 & 3 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right) \\ &= \frac{1}{4}\begin{bmatrix} -2 & 1 & 3 \\ 2 & 1 & -1 \\ 2 & -3 & -1 \end{bmatrix}. \end{aligned}$$

Example 5.2.6 **(A^k via Cayley–Hamilton)**

Let's illustrate how the Cayley–Hamilton theorem can be used to compute powers of the 3×3 matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ (which, incidentally, is **not** diagonalizable). The characteristic polynomial of A is $C_A(\lambda) = -\lambda^3 + 4\lambda^2 - 5\lambda + 2$. So

$$-A^3 + 4A^2 - 5A + 2I = 0.$$

Thus,

$$A^3 = 4A^2 - 5A + 2I.$$

Multiplying through by A and then re-using this expression for A^3, we get

$$A^4 = 4A^3 - 5A^2 + 2A = 4(4A^2 - 5A + 2I) - 5A^2 + 2A = 11A^2 - 18A + 8I.$$

Proceeding this way, we are able to express A^k as a linear combination of I, A and A^2.

At first sight this seems too tedious, but if you really think about it, you'll realize that we're effectively carrying out the Euclidean algorithm for polynomial division! More specifically, if we divide the polynomial λ^k by $C_A(\lambda)$, then we get

$$\lambda^k = q_k(\lambda)C_A(\lambda) + r_k(\lambda)$$

for some polynomials q_k and r_k (quotient and remainder). Plugging A into this, and using the Cayley–Hamilton theorem, we find that

$$A^k = r_k(A).$$

So all we have to do is find r_k. Fortunately, there are some fairly efficient software implementations of the Euclidean algorithm. Using one of these, we can find that r_{10} (the remainder of λ^{10} divided by $C_A(\lambda)$) is

$$r_{10}(\lambda) = 1013\lambda^2 - 2016\lambda + 1004.$$

Thus,

$$A^{10} = 1013A^2 - 2016A + 1004I = \begin{bmatrix} 1 & 10 & 3049 \\ 0 & 1 & 2046 \\ 0 & 0 & 1024 \end{bmatrix}.$$

5.3 Orthogonal and Unitary Diagonalization of Matrices

Let's return to our problem of trying to find a basis of orthogonal eigenvectors for a given $n \times n$ matrix. We begin by introducing some handy terminology. First, recall that a matrix $A \in M_{n\times n}(\mathbb{F})$ is diagonalizable over $\mathbb{F}$ if there is an invertible matrix $P \in M_{n\times n}(\mathbb{F})$ such that $P^{-1}AP$ is diagonal. This prompts the following.

Definition 5.3.1

Orthogonally Diagonalizable, Unitarily Diagonalizable

A matrix $A \in M_{n\times n}(\mathbb{R})$ is said to be **orthogonally diagonalizable** if there is an orthogonal matrix $Q \in M_{n\times n}(\mathbb{R})$ such that Q^TAQ is diagonal.

A matrix $A \in M_{n\times n}(\mathbb{C})$ is said to be **unitarily diagonalizable** if there is a unitary matrix $U \in M_{n\times n}(\mathbb{C})$ such that U^*AU is diagonal.

In Section 5.1, we proved:

Proposition 5.3.2 **(Criterion for Orthogonal and Unitary Diagonalizability)**

(a) A matrix $A \in M_{n\times n}(\mathbb{R})$ is orthogonally diagonalizable if and only if there is a basis for $\mathbb{R}^n$ consisting of orthonormal eigenvectors of A (orthonormal with respect to the standard inner product on $\mathbb{R}^n$, i.e., the dot product).

(b) A matrix $A \in M_{n\times n}(\mathbb{C})$ is unitarily diagonalizable if and only if there is a basis for $\mathbb{C}^n$ consisting of orthonormal eigenvectors for A (orthonormal with respect to the standard inner product on $\mathbb{C}^n$).

Now the burning question is: when is a matrix in $M_{n\times n}(\mathbb{R})$ orthogonally diagonalizable and when is a matrix in $M_{n\times n}(\mathbb{C})$ unitarily diagonalizable? The answers are surprisingly simple: precisely when the matrix is *symmetric* or *normal*, respectively.

The formal definitions are given below, but here is the key observation. If $A = UDU^*$ is unitarily diagonalizable then $A^* = (UDU^*)^* = UD^*U^*$. If D is diagonal, then so is D^*; if D is *real* too, then $D^* = D$, and so $A = A^*$. Thus, *if* a real matrix can be orthogonally diagonalized, it must be the case that $A = A^T$. On the other hand, if D has non-real entries, then we cannot say that $A = A^*$. However, notice that

$$AA^* = (UDU^*)(UD^*U^*) = UDD^*U^* = UD^*DU^* = A^*A.$$

Thus *if* a complex matrix can be unitarily diagonalized, it must be the case that $AA^* = A^*A$. This leads us to single out the following classes of matrices.

Definition 5.3.3 **Normal, Self-adjoint, Hermitian, Symmetric**

A matrix $A \in M_{n\times n}(\mathbb{F})$ is said to be **normal** if $AA^* = A^*A$.

A matrix $A \in M_{n\times n}(\mathbb{F})$ is said to be **self-adjoint** if $A = A^*$. If $A \in M_{n\times n}(\mathbb{C})$ is self-adjoint, then we say that A is **Hermitian**. If $A \in M_{n\times n}(\mathbb{R})$ is self-adjoint, then we say that A is **symmetric**.

That is, normal matrices are the matrices that commute with their adjoints. So, in particular, a self-adjoint matrix is normal. According to the definition, a symmetric matrix is Hermitian, but just like with unitary matrices, we reserve the adjective "Hermitian" for when we are discussing complex matrices.

Example 5.3.4

The matrices $A = \begin{bmatrix} 1 & 2-i \\ 2+i & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ 4 & 1 \end{bmatrix}$ are self-adjoint; A is Hermitian and B is symmetric.

The matrix $C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is normal but not self-adjoint. Indeed, $C^* = -C$ and therefore $CC^* = C^*C = -C^2$.

EXERCISE

Prove that if $A \in M_{n\times n}(\mathbb{F})$ is Hermitian, unitary or diagonal then A is normal.

With Schur's triangularization theorem (Theorem 5.2.1) in hand, we can easily prove the following result.

Theorem 5.3.5 **(Spectral Theorem for Hermitian Matrices)**

A square matrix in $M_{n\times n}(\mathbb{C})$ is Hermitian if and only if it is unitarily diagonalizable *and* if its eigenvalues are all real.

Proof: Suppose $A \in M_{n\times n}(\mathbb{C})$ is unitarily diagonalizable, say $A = UDU^*$ with U unitary and D diagonal. If the eigenvalues of A are real, then $D \in M_n(\mathbb{R})$. So $D^* = D^T = D$ since D is diagonal and real, and therefore

$$A^* = (UDU^*)^* = UDU^* = A.$$

This proves that A is Hermitian.

Conversely, suppose that A is Hermitian. By Schur's theorem, we know that there is a unitary matrix U such that $U^*AU = T$ where T is upper-triangular. Notice that

$$T^* = (U^*AU)^* = U^*A^*U = U^*AU = T$$

so T is Hermitian. Since T is also upper-triangular, it must be the case that T is diagonal. So A is unitarily diagonalizable. Furthermore, the entries on the diagonal of any Hermitian matrix are real (why?), so all of the eigenvalues of A are real, since they are the diagonal entries of T. □

EXERCISE

Prove the following fact that was used in the preceding proof. If $A \in M_{n\times n}(\mathbb{C})$ is Hermitian, then the diagonal entries of A are real.

If we specialize the above theorem to *real* Hermitian matrices, we arrive at the following result.

Theorem 5.3.6 **(Spectral Theorem for Symmetric Matrices)**

A square matrix in $M_{n\times n}(\mathbb{R})$ is symmetric if and only if it is orthogonally diagonalizable.

Proof: A matrix A in $M_{n\times n}(\mathbb{R})$ can be regarded as a matrix in $M_{n\times n}(\mathbb{C})$. In this light, we know from the preceding theorem that if $A \in M_{n\times n}(\mathbb{R})$ is symmetric (hence Hermitian) then there is a unitary matrix $U \in M_{n\times n}(\mathbb{C})$ and a real diagonal matrix $D \in M_{n\times n}(\mathbb{R})$ such that $A = UDU^*$. The matrix U was provided by Schur's theorem, which when A is real we know we can choose to be orthogonal. That is, we can find an orthogonal matrix U such that $A = UDU^* = UDU^T$. This proves that A is orthogonally diagonalizable.

We leave the proof of the converse as an easy exercise. □

EXERCISE

Complete the proof of Theorem 5.3.6 by showing that if $A \in M_{n\times n}(\mathbb{R})$ is orthogonally diagonalizable then A is symmetric.

It is important to note that in the spectral theorem for Hermitian matrices the requirement that the eigenvalues are all real is definitely required, as shown in the next example.

Example 5.3.7 Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. We find the eigenvalues of A are i and $-i$ with bases for the eigenspaces given by

$$\left\{ \begin{bmatrix} i \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad \left\{ \begin{bmatrix} -i \\ 1 \end{bmatrix} \right\},$$

respectively. Notice that with the standard inner product on $\mathbb{C}^2$ we have

$$\left\langle \begin{bmatrix} i \\ 1 \end{bmatrix}, \begin{bmatrix} -i \\ 1 \end{bmatrix} \right\rangle = i^2 + 1 = 0.$$

So the eigenvectors are orthogonal. Normalizing them and putting them in a matrix gives us the unitary matrix

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} i & -\frac{1}{\sqrt{2}} i \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Then, since both the columns of U are eigenvectors of A, we have $U^*AU = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$.

So we see here that A is unitarily diagonalizable, but the eigenvalues are not real, and A is not Hermitian. (However, notice that A is normal!)

Here is the definitive result that says when a matrix can be unitarily diagonalized.

Theorem 5.3.8 **(Spectral Theorem for Normal Matrices)**

A square matrix in $M_{n \times n}(\mathbb{C})$ is normal if and only if it is unitarily diagonalizable.

Proof: Suppose that $A \in M_{n \times n}(\mathbb{F})$ is normal. By Schur's triangularization theorem, we can find a unitary matrix U and an upper-triangular matrix T such that $U^*AU = T$. Then

$$TT^* = (U^*AU)(U^*A^*U) = U^*AA^*U = U^*A^*AU = T^*T.$$

Let t_{ij} denote the (i,j)th entry of T. Then the $(1,1)$ entry of TT^* is

$$t_{11}\overline{t_{11}} + t_{12}\overline{t_{12}} + \cdots + t_{1n}\overline{t_{1n}} = |t_{11}|^2 + |t_{12}|^2 + \cdots + |t_{1n}|^2.$$

On the other hand, the $(1,1)$ entry of T^*T is $|t_{11}|^2$. It follows that $t_{12} = \cdots = t_{1n} = 0$. Next, comparing the $(2,2)$ entries in a similar manner, we find that $t_{21} = t_{23} = t_{24} = \cdots = t_{2n} = 0$. Continuing in this way, we see that T must be diagonal.

Thus, if A is normal, it is unitarily diagonalizable. The converse is left as an exercise. □

EXERCISE

Complete the proof of Theorem 5.3.8 by showing that if $A \in M_{n \times n}(\mathbb{F})$ is unitarily diagonalizable then A is normal.

To summarize: we now know that if we have a normal (e.g., Hermitian) matrix A, then it is diagonalizable (over $\mathbb{C}$) with a basis of orthonormal eigenvectors; further, if A is real and symmetric, then this can be done over $\mathbb{R}$. Unfortunately, the results so far don't really give any indication as to how we can find this basis of orthogonal eigenvectors—or equivalently, how we can find the unitary U or orthogonal Q that diagonalizes A. It is tempting to simply find a basis of eigenvectors as usual (that is, diagonalize as usual) and perform Gram–Schmidt on that basis to obtain an orthonormal basis. There's only one problem with this idea: there's a chance that while performing Gram–Schmidt, you are no longer left with eigenvectors!

Example 5.3.7 gives us a hint as to how to get around this. Notice that the two eigenvectors we found were already orthogonal. This was not a coincidence! For normal matrices (so in particular for Hermitian and symmetric matrices), eigenvectors corresponding to different eigenvalues are orthogonal. We prove this, along with a few other useful facts about normal matrices, in the following proposition.

Proposition 5.3.9 **(Properties of Normal Matrices)**

Let $A \in M_{n\times n}(\mathbb{F})$ be a normal matrix. Equip $\mathbb{F}^n$ with the standard inner product. Then:

(a) For all $\vec{x} \in \mathbb{F}^n$, $\|A\vec{x}\| = \|A^*\vec{x}\|$.

(b) If $\vec{x} \in \mathbb{F}^n$ is an eigenvector for A with eigenvalue λ, then $\vec{x}$ is an eigenvector for A^* with eigenvalue $\overline{\lambda}$.

(c) If $\vec{x}$ and $\vec{y}$ in $\mathbb{F}^n$ are eigenvectors of A with *distinct* eigenvalues λ and μ, then $\vec{x}$ is orthogonal to $\vec{y}$.

Proof: (a) We have

$$\|A\vec{x}\|^2 = (A\vec{x})^*(A\vec{x}) = \vec{x}^*A^*A\vec{x} = \vec{x}^*AA^*\vec{x} = (A^*\vec{x})^*(A^*\vec{x}) = \|A^*\vec{x}\|^2.$$

(b) Suppose that $A\vec{x} = \lambda\vec{x}$. We want to prove that $A^*\vec{x} = \overline{\lambda}\vec{x}$. It will suffice to show that $\|A^*\vec{x} - \overline{\lambda}\vec{x}\| = 0$. Now, by part (a), we have

$$\|A^*\vec{x} - \overline{\lambda}\vec{x}\| = \|(A^* - \overline{\lambda}I)\vec{x}\| = \|(A^* - \overline{\lambda}I)^*\vec{x}\| = \|(A - \lambda I)\vec{x}\|.$$

Since $A\vec{x} = \lambda\vec{x}$, the last term above is 0, completing the proof.

(c) We want to show that $\langle \vec{x}, \vec{y}\rangle = 0$. The trick is to consider $\lambda \langle \vec{x}, \vec{y}\rangle$:

$$\begin{aligned}
\lambda \langle \vec{x}, \vec{y}\rangle &= \langle \lambda\vec{x}, \vec{y}\rangle \\
&= \langle A\vec{x}, \vec{y}\rangle \\
&= \vec{y}^*(A\vec{x}) \\
&= (A^*\vec{y})^*\vec{x} \\
&= \langle \vec{x}, A^*\vec{y}\rangle \\
&= \langle \vec{x}, \overline{\mu}\vec{y}\rangle \qquad \text{(by part (b))} \\
&= \mu \langle \vec{x}, \vec{y}\rangle .
\end{aligned}$$

Thus $(\lambda - \mu)\langle \vec{x}, \vec{y}\rangle = 0$. Since $\lambda \neq \mu$, it follows that $\langle \vec{x}, \vec{y}\rangle = 0$, as required.

□

Proposition 5.3.9(c) should remind you of Lemma 3.2.10, which says that eigenvectors corresponding to distinct eigenvalues are linearly independent. Here we have that when A is normal, eigenvectors corresponding to distinct eigenvalues are orthogonal. It's important to note that eigenvectors corresponding to the same eigenvalues need not be orthogonal. (Consider then eigenvector $\vec{x}$ and its scalar multiple $2\vec{x}$ for example.)

So, with this in mind, we now know that we don't have to perform Gram–Schmidt on the *entire* basis of eigenvectors, just on the basis for each eigenspace! This leads to the following algorithm to unitarily diagonalize a normal matrix (so, in particular, a Hermitian or symmetric matrix).

ALGORITHM (Unitary Diagonalization of a Normal Matrix)

To unitarily diagonalize a normal matrix $A \in M_{n\times n}(\mathbb{F})$:

1. Diagonalize A as usual (see the Algorithm on page 59), obtaining $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ and a basis of eigenvectors for $\mathbb{F}^n$.
2. Perform the Gram–Schmidt procedure on the bases of each of the eigenspaces E_{λ_i} of A, obtaining orthonormal bases for the eigenspaces. (This step is to be carried out using the standard inner product on $\mathbb{F}^n$.)
3. Take the union of the orthonormal bases from Step 2 to obtain $\mathcal{D} = \{\vec{w}_1, \ldots, \vec{w}_n\}$, which is an orthonormal basis of eigenvectors for $\mathbb{F}^n$. Order the basis $\mathcal{D}$ so that the orthonormal basis for E_{λ_1} is followed by the orthonormal basis for E_{λ_2}, etc.
4. Let $U = \begin{bmatrix} \vec{w}_1 & \cdots & \vec{w}_n \end{bmatrix}$. Then U is unitary and $U^*AU = D$, with D as in Step 1.

If the above process is carried out on a symmetric matrix $A \in M_{n\times n}(\mathbb{R})$, then the resulting matrix U will be real and orthogonal, and the diagonal matrix D will have real entries. Thus, we will have $U^TAU = D$ with the columns of U forming an orthonormal basis for $\mathbb{R}^n$.

In particular, suppose an $n\times n$ Hermitian (or symmetric) matrix A has n distinct eigenvalues. Since A has distinct eigenvalues, we know it's diagonalizable, and moreover, any basis of eigenvectors for $\mathbb{F}^n$ will consist of one eigenvector for each eigenvalue. Since A is Hermitian, hence normal, this basis must be orthogonal, since it consists of eigenvectors corresponding to distinct eigenvalues. In this case, to obtain an orthonormal basis (and therefore a unitary U such that U^*AU is diagonal) we simply need to normalize the eigenvectors in this basis.

Here is an example demonstrating this special case.

Example 5.3.10

Let $A = \begin{bmatrix} 1 & 1-i \\ 1+i & 0 \end{bmatrix}$. Notice that $A^* = A$ so A is Hermitian. We find that the characteristic polynomial of A is $C_A(\lambda) = (\lambda+1)(\lambda-2)$, so A has the distinct eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 2$. (As an aside, observe that these eigenvalues are real, as predicted by the spectral theorem for Hermitian matrices.)

So already we know that A is diagonalizable. Let's find eigenvectors. For λ_1, we row reduce

$$A - \lambda_1 I = \begin{bmatrix} 2 & 1-i \\ 1+i & 1 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2}(1-i) \\ 0 & 0 \end{bmatrix}.$$

(for this computation, use $(1+i)^{-1} = \frac{1}{2}(1-i)$). Therefore a basis for the eigenspace corresponding to λ_1 is $\left\{ \begin{bmatrix} -\frac{1}{2} + \frac{1}{2}i \\ 1 \end{bmatrix} \right\}$, or more simply $\left\{ \begin{bmatrix} -1+i \\ 2 \end{bmatrix} \right\}$.

Next, for λ_2, we row reduce

$$A - \lambda_2 i = \begin{bmatrix} -1 & 1-i \\ 1+i & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -1+i \\ 0 & 0 \end{bmatrix}.$$

So a basis for the corresponding eigenspace is $\left\{ \begin{bmatrix} 1-i \\ 1 \end{bmatrix} \right\}$.

Putting these two bases together, we get the basis $\left\{ \begin{bmatrix} -1+i \\ 2 \end{bmatrix}, \begin{bmatrix} 1-i \\ 1 \end{bmatrix} \right\}$ for $\mathbb{C}^2$. Notice that this basis is orthogonal, since

$$\left\langle \begin{bmatrix} -1+i \\ 2 \end{bmatrix}, \begin{bmatrix} 1-i \\ 1 \end{bmatrix} \right\rangle = (-1+i)\overline{(1-i)} + 2\overline{(1)} = -2+2 = 0,$$

exactly as predicted by Proposition 5.3.9(c). To get an *orthonormal* basis, we simply normalize, obtaining

$$\left\{ \frac{1}{\sqrt{6}} \begin{bmatrix} -1+i \\ 2 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} 1-i \\ 1 \end{bmatrix} \right\}.$$

And now here is an example of the algorithm in all its glory.

Example 5.3.11 Let's unitarily diagonalize the matrix

$$A = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}.$$

First note that A is symmetric. A quick computation gives $C_A(\lambda) = \lambda(9-\lambda)^2$. We'll now find bases for the eigenspaces corresponding to $\lambda = 0$ and $\lambda = 9$. Since A is Hermitian, we know it's diagonalizable so we should have the geometric multiplicities $g_0 = 1$ and $g_9 = 2$.

For $\lambda = 0$, we row reduce

$$A - 0I = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore a basis for the eigenspace corresponding to 0 is $\left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\}$. We wish to have an orthonormal basis for this eigenspace, so we normalize and choose instead the basis

$\left\{ \frac{1}{3} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\}$. To find the eigenspace corresponding to $\lambda = 9$ we row reduce

$$A - 9I = \begin{bmatrix} -4 & -4 & -2 \\ -4 & -4 & -2 \\ -2 & -2 & -1 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

giving a basis $\{\vec{v}_1, \vec{v}_2\} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}$. Note that $\vec{v}_1$ and $\vec{v}_2$ are not orthogonal, so we must perform Gram-Schmidt to obtain an orthogonal basis for this eigenspace.

Doing so, we let $\vec{w}_1 = \vec{v}_1$ and

$$\vec{w}_2 = \vec{v}_2 - \frac{\langle \vec{v}_2, \vec{w}_1 \rangle}{\|\vec{w}_1\|^2} \vec{w}_1 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix}.$$

Now $\{\vec{w}_1, \vec{w}_2\}$ is an orthogonal basis for the eigenspace corresponding to $\lambda = 9$. To obtain an orthonormal basis $\{\vec{u}_1, \vec{u}_2\}$ we set

$$\vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{u}_2 = \frac{1}{\sqrt{18}} \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix}.$$

Finally, $U^*AU = D$ where

$$U = \begin{bmatrix} \frac{2}{3} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{18}} \\ \frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{18}} \\ \frac{1}{3} & 0 & \frac{4}{\sqrt{18}} \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}.$$

As usual, the order of diagonal entries in D matches the order of eigenvectors in our change-of-basis matrix U.

EXERCISE

A matrix $A \in M_{n \times n}(\mathbb{C})$ is said to be **skew-Hermitian** if $A^* = -A$.

(a) Prove that a skew-Hermitian matrix is unitarily diagonalizable.

(b) Let $A = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$. Find a unitary matrix U and a diagonal matrix D such that $U^*AU = D$.

5.4 Inner Products and Hermitian Matrices

We now know how to determine if an $n \times n$ matrix A admits a basis for $\mathbb{F}^n$ consisting of orthonormal eigenvectors—at least if $\mathbb{F}^n$ is equipped with the standard inner product. But what if $\mathbb{F}^n$ is equipped with some other inner product? The goal of this section will be to address this question. We'll begin by trying to describe what all inner products on $\mathbb{F}^n$ (and indeed on any finite-dimensional inner product space) look like in terms of matrices.

5.4.1 The Matrix of an Inner Product

Suppose we have an inner product $\langle\ ,\ \rangle$ on $\mathbb{C}^2$ and suppose we know what it does to the standard basis $\{\vec{e}_1, \vec{e}_2\}$:

$$\langle \vec{e}_1, \vec{e}_1\rangle = 2, \quad \langle \vec{e}_2, \vec{e}_2\rangle = 3, \quad \text{and} \quad \langle \vec{e}_1, \vec{e}_2\rangle = 1+i.$$

Is it possible to recover the entire inner product just from this information? The answer is yes! Given $\vec{v}, \vec{u} \in \mathbb{C}^2$, we can express them uniquely in terms of the standard basis as

$$\vec{v} = a\vec{e}_1 + b\vec{e}_2 \quad \text{and} \quad \vec{u} = c\vec{e}_1 + d\vec{e}_2 \qquad (a,b,c,d \in \mathbb{C}).$$

Then

$$\begin{aligned}\langle \vec{v}, \vec{u}\rangle &= \langle a\vec{e}_1 + b\vec{e}_2, c\vec{e}_1 + d\vec{e}_2\rangle \\ &= \langle a\vec{e}_1, c\vec{e}_1\rangle + \langle b\vec{e}_2, c\vec{e}_1\rangle + \langle a\vec{e}_1, d\vec{e}_2\rangle + \langle b\vec{e}_2, d\vec{e}_2\rangle \\ &= a\overline{c}\,\langle \vec{e}_1, \vec{e}_1\rangle + b\overline{c}\,\langle \vec{e}_2, \vec{e}_1\rangle + a\overline{d}\,\langle \vec{e}_1, \vec{e}_2\rangle + b\overline{d}\,\langle \vec{e}_2, \vec{e}_2\rangle \\ &= 2a\overline{c} + (1-i)b\overline{c} + (1+i)a\overline{d} + 3b\overline{d}.\end{aligned}$$

This completely defines our inner product. So, just like linear maps are determined by what they do to basis vectors, it appears, at least in this example, that inner products may be determined by what they do to basis vectors.

Recall that the fact that linear maps are determined by the images of the basis vectors is intimately tied to the fact that every linear map can be represented by a matrix (once we've chosen bases for our vector spaces of course). So a natural question arises: can we represent an inner product by a matrix, and does every matrix represent an inner product?

Continuing the example above, let $\mathcal{B}$ be the standard basis for $\mathbb{C}^2$. Then $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} a \\ b \end{bmatrix}$, $[\vec{u}]_{\mathcal{B}} = \begin{bmatrix} c \\ d \end{bmatrix}$ and (forgive us for pulling this identity out of thin air)

$$\begin{bmatrix} \overline{c} & \overline{d} \end{bmatrix} \begin{bmatrix} 2 & 1-i \\ 1+i & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a\overline{c} + (1-i)b\overline{c} + (1+i)a\overline{d} + 3b\overline{d} \end{bmatrix}.$$

Hmm! If we identify a 1×1 matrix with an element of $\mathbb{C}$, the above equation can be written as

$$\overline{[\vec{u}]_{\mathcal{B}}^T}\, A\, [\vec{v}]_{\mathcal{B}} = \langle \vec{v}, \vec{u}\rangle$$

for some 2×2 matrix A.

Does this type of expression seem familiar? You might recall that the standard inner product on $\mathbb{C}^2$ is given by

$$\langle \vec{v}, \vec{u}\rangle = \overline{\vec{u}^T}\vec{v} = \overline{\vec{u}^T}\ I\ \vec{v}.$$

Comparing both equations, the difference appears to be that we changed coordinates from standard to $\mathcal{B}$-coordinates, and changed the identity matrix to this peculiar matrix A.

Let's phrase our motivating question a little better. Let V be an n-dimensional vector space with basis $\mathcal{B}$.

1. Let $\langle\ ,\ \rangle$ be an inner product on V. Is there an $n \times n$ matrix A such that
$$\langle \vec{v}, \vec{w} \rangle = [\vec{w}]_{\mathcal{B}}^* \ A \ [\vec{v}]_{\mathcal{B}} \quad \text{for all } \vec{v}, \vec{w} \in V?$$

2. Let A be an $n \times n$ matrix. Does
$$\langle \vec{v}, \vec{w} \rangle = [\vec{w}]_{\mathcal{B}}^* \ A \ [\vec{v}]_{\mathcal{B}}$$
define an inner product on V?

To try and get some headway into the problem, let's assume that $\dim(V) = 2$ say with basis $\mathcal{B} = \{\vec{g}_1, \vec{g}_2\}$. Let $\vec{v} = a\vec{g}_1 + b\vec{g}_2$ and $\vec{w} = c\vec{g}_1 + d\vec{g}_2$ be arbitrary vectors in V, so that $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $[\vec{w}]_{\mathcal{B}} = \begin{bmatrix} c \\ d \end{bmatrix}$.

Suppose there is a 2×2 matrix $A = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$ such that $\langle \vec{v}, \vec{w} \rangle = [\vec{w}]_{\mathcal{B}}^* \ A \ [\vec{v}]_{\mathcal{B}}$. Then
$$\langle \vec{v}, \vec{w} \rangle = \begin{bmatrix} \bar{c} & \bar{d} \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a\bar{c}w + a\bar{d}y + b\bar{c}x + b\bar{d}z \end{bmatrix}.$$

Since
$$\langle \vec{v}, \vec{w} \rangle = a\bar{c}\,\langle \vec{g}_1, \vec{g}_1 \rangle + a\bar{d}\,\langle \vec{g}_1, \vec{g}_2 \rangle + b\bar{c}\,\langle \vec{g}_2, \vec{g}_1 \rangle + b\bar{d}\,\langle \vec{g}_2, \vec{g}_2 \rangle$$
we must have
$$a\bar{c}w + a\bar{d}y + b\bar{c}x + b\bar{d}z = a\bar{c}\,\langle \vec{g}_1, \vec{g}_1 \rangle + a\bar{d}\,\langle \vec{g}_1, \vec{g}_2 \rangle + b\bar{c}\,\langle \vec{g}_2, \vec{g}_1 \rangle + b\bar{d}\,\langle \vec{g}_2, \vec{g}_2 \rangle$$
for all $a, b, c, d \in \mathbb{F}$. Thus,
$$\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} \langle \vec{g}_1, \vec{g}_1 \rangle & \langle \vec{g}_2, \vec{g}_1 \rangle \\ \langle \vec{g}_1, \vec{g}_2 \rangle & \langle \vec{g}_2, \vec{g}_2 \rangle \end{bmatrix}.$$
(For instance, if we let $b = d = 0$ and $a = c = 1$, then we find that $w = \langle \vec{g}_1, \vec{g}_1 \rangle$. Similar choices allow us to determine x, y and z.)

For the general case of an n-dimensional inner product space with basis $\mathcal{B} = \{\vec{g}_1, \ldots, \vec{g}_n\}$, if there is to be an $n \times n$ matrix $A = [A_{ij}]$ such that $\langle \vec{v}, \vec{w} \rangle = [\vec{w}]_{\mathcal{B}}^* A [\vec{v}]_{\mathcal{B}}$ then we must have $A_{ij} = \langle \vec{g}_j, \vec{g}_i \rangle$. Furthermore, since $\langle \vec{v}, \vec{w} \rangle = \overline{\langle \vec{w}, \vec{v} \rangle}$ it follows that $A_{ij} = \overline{A_{ji}}$ (take $\vec{v} = \vec{g}_i$ and $\vec{w} = \vec{g}_j$). That is, it must be the case that $A = A^*$ is self-adjoint. What other features must A have?

Theorem 5.4.1 **(Characterization of Inner Products in Terms of Matrices)**

Let V be a vector space over $\mathbb{F}$ with basis $\mathcal{B} = \{\vec{g}_1, \ldots, \vec{g}_n\}$, and let $A \in M_{n \times n}(\mathbb{F})$. Then
$$\langle \vec{v}, \vec{w} \rangle = [\vec{w}]_{\mathcal{B}}^* \ A \ [\vec{v}]_{\mathcal{B}} \qquad (\vec{v}, \vec{w} \in V)$$
defines an inner product on V if and only if $A = A^*$ and all the eigenvalues of A are positive.

Conversely, if $\langle\ ,\ \rangle$ is an inner product of V, then there is a self-adjoint matrix A such that
$$\langle \vec{v}, \vec{w} \rangle = [\vec{w}]_{\mathcal{B}}^* \ A \ [\vec{v}]_{\mathcal{B}} \quad \text{for all } \vec{v}, \vec{w} \in V.$$

Explicitly, this matrix is given by
$$A = \begin{bmatrix} \langle \vec{g}_1, \vec{g}_1 \rangle & \langle \vec{g}_2, \vec{g}_1 \rangle & \cdots & \langle \vec{g}_n, \vec{g}_1 \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \vec{g}_1, \vec{g}_n \rangle & \langle \vec{g}_2, \vec{g}_n \rangle & \cdots & \langle \vec{g}_n, \vec{g}_n \rangle \end{bmatrix}.$$

Proof: First, assume that $\langle \vec{v}, \vec{w} \rangle = [\vec{w}]_{\mathcal{B}}^* A [\vec{v}]_{\mathcal{B}}$ defines an inner product on V. Then it must satisfy the inner product axioms:

1. $\langle \vec{v}, \vec{w} \rangle = \overline{\langle \vec{w}, \vec{v} \rangle}$.
2. $\langle \alpha \vec{v}, \vec{w} \rangle = \alpha \langle \vec{v}, \vec{w} \rangle$.
3. $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$.
4. (a) $\langle \vec{v}, \vec{v} \rangle \geq 0$.
   (b) If $\langle \vec{v}, \vec{v} \rangle = 0$ then $\vec{v} = \vec{0}$.

The discussion preceding the statement of the theorem shows that we must have $A = A^*$. For the claim about the eigenvalues, suppose that $A[\vec{x}]_{\mathcal{B}} = \lambda [\vec{x}]_{\mathcal{B}}$ with $[\vec{x}]_{\mathcal{B}} \neq \vec{0}$ an eigenvector of A. Then by multiplying both sides by $[\vec{x}]_{\mathcal{B}}^*$ we get

$$[\vec{x}]_{\mathcal{B}}^* A [\vec{x}]_{\mathcal{B}} = \lambda [\vec{x}]_{\mathcal{B}}^* [\vec{x}]_{\mathcal{B}}.$$

The left-side is $\langle \vec{x}, \vec{x} \rangle$ which by axiom 4 is > 0 since $\vec{x} \neq \vec{0}$. The number $[\vec{x}]_{\mathcal{B}}^* [\vec{x}]_{\mathcal{B}}$ on the right-side is the standard inner product of $[\vec{x}]_{\mathcal{B}}$ with itself, hence must also be > 0. Thus

$$\lambda = \frac{\langle \vec{x}, \vec{x} \rangle}{[\vec{x}]_{\mathcal{B}}^* [\vec{x}]_{\mathcal{B}}}$$

is positive.

Conversely, assume that $A = A^*$ and that the eigenvalues of A are positive. We want to prove that $\langle \vec{v}, \vec{w} \rangle = [\vec{w}]_{\mathcal{B}}^* A [\vec{v}]_{\mathcal{B}}$ satisfies the inner product axioms above. To show that axiom 1 is satisfied, we will "un-abuse" notation and revert back to viewing $[\vec{w}]_{\mathcal{B}}^* A [\vec{v}]_{\mathcal{B}}$ as the 1×1 matrix $[\langle \vec{v}, \vec{w} \rangle]$. Then

$$\begin{aligned}
[\langle \vec{v}, \vec{w} \rangle]^* &= ([\vec{w}]_{\mathcal{B}}^* A [\vec{v}]_{\mathcal{B}})^* \\
&= [\vec{v}]_{\mathcal{B}}^* A^* [\vec{w}]_{\mathcal{B}} \\
&= [\vec{v}]_{\mathcal{B}}^* A [\vec{w}]_{\mathcal{B}} \qquad (\text{since } A = A^*) \\
&= [\langle \vec{w}, \vec{v} \rangle]
\end{aligned}$$

which, since the transpose of a 1×1 matrix is just the matrix itself, implies $\overline{\langle \vec{v}, \vec{w} \rangle} = \langle \vec{w}, \vec{v} \rangle$.

For the other axioms we resume our abuse of notation and identify the 1×1 matrix $[\langle \vec{v}, \vec{w} \rangle]$ with the number $\langle \vec{v}, \vec{w} \rangle$. We have

$$\begin{aligned}
\langle \vec{v} + \vec{u}, \vec{w} \rangle &= [\vec{w}]_{\mathcal{B}}^* A [\vec{v} + \vec{u}]_{\mathcal{B}} \\
&= [\vec{w}]_{\mathcal{B}}^* A ([\vec{v}]_{\mathcal{B}} + [\vec{u}]_{\mathcal{B}}) \\
&= [\vec{w}]_{\mathcal{B}}^* A [\vec{v}]_{\mathcal{B}} + [\vec{w}]_{\mathcal{B}}^* A [\vec{u}]_{\mathcal{B}} \\
&= \langle \vec{v}, \vec{w} \rangle + \langle \vec{u}, \vec{w} \rangle .
\end{aligned}$$

Similarly,

$$\begin{aligned}
\langle \alpha \vec{v}, \vec{w} \rangle &= [\vec{w}]_{\mathcal{B}}^* A [\alpha \vec{v}]_{\mathcal{B}} \\
&= \alpha [\vec{w}]_{\mathcal{B}}^* A [\vec{v}]_{\mathcal{B}} \\
&= \alpha \langle \vec{v}, \vec{w} \rangle .
\end{aligned}$$

As usual, the interesting situation is axiom 4. It's not clear what conditions we need on A so that $[\vec{v}]_{\mathcal{B}}^* A[\vec{v}]_{\mathcal{B}} \geq 0$.

Since $A^* = A$, i.e., A is Hermitian, the spectral theorem (Theorem 5.3.5) tells us there is a unitary matrix $U \in M_{n\times n}(\mathbb{C})$ and a diagonal matrix $D \in M_{n\times n}(\mathbb{R})$ so that $U^*AU = D$. Then

$$\langle \vec{v}, \vec{w} \rangle = [\vec{w}]_{\mathcal{B}}^* UDU^*[\vec{v}]_{\mathcal{B}} = (U^*[\vec{w}]_{\mathcal{B}})^* D(U^*[\vec{v}]_{\mathcal{B}}).$$

Since U^* is invertible, we can think of it as a change of basis matrix. In fact, we have $U^*[\vec{v}]_{\mathcal{B}} = [\vec{v}]_{\mathcal{C}}$ for all $\vec{v} \in V$, where $\mathcal{C}$ is some other basis (it's not important what it is). So we now have

$$\langle \vec{v}, \vec{w} \rangle = [\vec{w}]_{\mathcal{C}}^* D[\vec{v}]_{\mathcal{C}}.$$

Remember that D is a diagonal matrix with entries along the diagonal equal to the eigenvalues of A, and these eigenvalues are real and positive (by assumption). So if

$$[\vec{w}]_{\mathcal{C}} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}, \quad [\vec{v}]_{\mathcal{C}} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}.$$

Then

$$\langle \vec{v}, \vec{w} \rangle = [\vec{w}]_{\mathcal{C}}^* D[\vec{v}]_{\mathcal{C}} = \lambda_1 v_1 \overline{w}_1 + \cdots + \lambda_n v_n \overline{w}_n.$$

In particular,

$$\langle \vec{v}, \vec{v} \rangle = \lambda_1 |v_1|^2 + \cdots + \lambda_n |v_n|^2.$$

Therefore $\langle \vec{v}, \vec{v} \rangle \geq 0$ since $\lambda_i > 0$ for all i. Furthermore, if $\langle \vec{v}, \vec{v} \rangle = 0$ then necessarily $|v_i|^2 = 0$ for all i, and consequently $\vec{v} = \vec{0}$. This completes the proof that $\langle \vec{v}, \vec{w} \rangle$ is an inner product.

The final assertion in the theorem was proved in the discussion immediately preceding the statement of the theorem. ☐

Let's give the matrix A in the previous theorem a name.

Definition 5.4.2
Gram Matrix

Let V be a finite-dimensional inner product space over $\mathbb{F}$ with inner product $\langle\ ,\ \rangle$ and basis $\mathcal{B} = \{\vec{g}_1, \ldots, \vec{g}_n\}$. The **Gram matrix of $\langle\ ,\ \rangle$ with respect to $\mathcal{B}$** is the matrix whose (i, j)th entry is $A_{ij} = \langle \vec{g}_j, \vec{g}_i \rangle$. That is,

$$A = \begin{bmatrix} \langle \vec{g}_1, \vec{g}_1 \rangle & \cdots & \langle \vec{g}_n, \vec{g}_1 \rangle \\ \vdots & \ddots & \vdots \\ \langle \vec{g}_1, \vec{g}_n \rangle & \cdots & \langle \vec{g}_n, \vec{g}_n \rangle \end{bmatrix}.$$

Theorem 5.4.1 shows that the Gram matrix can be used to compute the inner product via the formula

$$\langle \vec{v}, \vec{w} \rangle = [\vec{w}]_{\mathcal{B}}^* A[\vec{v}]_{\mathcal{B}}.$$

Example 5.4.3 In $\mathbb{F}^n$ with the standard inner product, if we let $\mathcal{B} = \{\vec{e}_1, \ldots, \vec{e}_n\}$ be the standard basis, then the corresponding Gram matrix is

$$\begin{bmatrix} \langle \vec{e}_1, \vec{e}_1 \rangle & \cdots & \langle \vec{e}_n, \vec{e}_1 \rangle \\ \vdots & \ddots & \vdots \\ \langle \vec{e}_1, \vec{e}_n \rangle & \cdots & \langle \vec{e}_n, \vec{e}_n \rangle \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix},$$

i.e. it's the $n \times n$ identity matrix! This re-establishes our familiar formula

$$\langle \vec{v}, \vec{w} \rangle = \vec{w}^* I_n \vec{v} = \vec{w}^* \vec{v}$$

for the standard inner product on $\mathbb{F}^n$.

More generally, if V is any n-dimensional inner product space, and if $\mathcal{B}$ is an *orthonormal* basis for V, then the corresponding Gram matrix is the $n \times n$ identity matrix. This means that we can compute the inner product on V as though it were the standard inner product on $\mathbb{F}^n$ (once we convert to $\mathcal{B}$-coordinates):

$$\langle \vec{v}, \vec{w} \rangle = [\vec{w}]_{\mathcal{B}}^* [\vec{v}]_{\mathcal{B}}.$$

Another reason why orthonormal bases are nice!

EXERCISE

Let V be an n-dimensional inner product space and let $\mathcal{B}$ be an *orthonormal* basis for V.

(a) Show that the Gram matrix with respect to $\mathcal{B}$ is I_n.

(b) What does the Gram matrix with respect to an orthogonal basis look like?

As you may have experienced, sometimes it is difficult to prove or disprove that a potential inner product satisfies axiom 4 (positive-definiteness) of Definition 4.1.1. Theorem 5.4.1 gives us an algorithmic way to check. We simply choose a basis, compute the corresponding Gram matrix A, check that A is Hermitian, verify that A actually performs the inner product for us, and then determine if the eigenvalues of A are all positive.

Example 5.4.4 Is

$$\langle a + bx, c + dx \rangle = 2a\bar{c} + (1 + i)b\bar{c} + (1 - i)a\bar{d} + 3b\bar{d}$$

an inner product on $\mathcal{P}_1(\mathbb{C})$? Let's find out.

Let $\mathcal{B}$ be the standard basis for $\mathcal{P}_1(\mathbb{C})$. The corresponding Gram matrix is given by

$$A = \begin{bmatrix} \langle 1, 1 \rangle & \langle x, 1 \rangle \\ \langle 1, x \rangle & \langle x, x \rangle \end{bmatrix} = \begin{bmatrix} 2 & 1 + i \\ 1 - i & 3 \end{bmatrix}.$$

First thing we need to check is that this matrix is Hermitian, which it is. Now we need to check that it actually performs the inner product for us. That is, we need to check if

$\langle p, q\rangle = [q]^*_{\mathcal{B}} A [p]_{\mathcal{B}}$. We have

$$[c + dx]^*_{\mathcal{B}} A [a + bx]_{\mathcal{B}} = \begin{bmatrix} \overline{c} & \overline{d} \end{bmatrix} \begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a\overline{c} + (1+i)b\overline{c} + (1-i)a\overline{d} + 3b\overline{d} \end{bmatrix}.$$

Therefore this matrix does the trick! So, to check whether or not it's an inner product, we need to compute the eigenvalues and make sure they're all positive. We have

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 2-\lambda & 1+i \\ 1-i & 3-\lambda \end{vmatrix} \\ &= (2-\lambda)(3-\lambda) - (1+i)(1-i) \\ &= \lambda^2 - 5\lambda + 4 \\ &= (\lambda - 4)(\lambda - 1). \end{aligned}$$

Since the eigenvalues are 1 and 4, which are both positive, Theorem 5.4.1 allows us to conclude that this is indeed an inner product.

Example 5.4.5 Let's determine whether or not

$$\left\langle \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \right\rangle = 2v_1w_1 + 2v_2w_2 + 2v_3w_3 - v_1w_2 - v_2w_1 - v_2w_3 - v_3w_2$$

is an inner product on $\mathbb{R}^3$.

Let $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ be the standard basis for $\mathbb{R}^3$. Our candidate matrix for this potential inner product is

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

This matrix is certainly Hermitian, which is a good start. Furthermore we have

$$\begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 2v_1w_1 + 2v_2w_2 + 2v_3w_3 - v_1w_2 - v_2w_1 - v_2w_3 - v_3w_2 \end{bmatrix}$$

so the matrix does actually perform the inner product for us. Finally, you can compute its eigenvalues and get them to be $\{2, 2+\sqrt{2}, 2-\sqrt{2}\}$, all of which are positive! Therefore this is an inner product.

EXERCISE

Determine whether

$$\langle \vec{z}, \vec{w}\rangle = z_1\overline{w_1} - iz_1\overline{w_2} + iz_2\overline{w_1}$$

is an inner product on $\mathbb{C}^2$.

5.4.2 Unitary Diagonalization of Operators

We're finally ready to solve the problem that was stated in the beginning of this chapter. Let $L\colon V \to V$ be a linear operator on an n-dimensional inner product over $\mathbb{F}$. When does V admit an orthonormal basis of eigenvectors for L?

If $V = \mathbb{F}^n$ with the standard inner product, then we answered this question in Section 5.3. Consider the standard matrix $A = [L]_{\mathcal{S}}$ of L. Then, if $\mathbb{F} = \mathbb{C}$, V will have an orthonormal basis of eigenvectors if and only if A is unitarily diagonalizable; while if $\mathbb{F} = \mathbb{R}$, V will admit an orthonormal basis of eigenvectors if and only if A is orthogonally diagonalizable. The key to figuring out when this is the case was the adjoint (conjugate-transpose) A^* of A. We proved in Theorems 5.3.8 and 5.3.6 that a complex square matrix A will be unitarily diagonalizable if and only if A is normal, and a real square matrix A will be orthogonally diagonalizable if and only if A is symmetric.

Now what about general inner product spaces? We can again pick a basis $\mathcal{B}$ and consider the matrix $A = [L]_{\mathcal{B}}$. Then we're in search of a basis for $\mathbb{F}^n$ consisting of orthonormal eigenvectors of A. We know how to do this, right? *Not quite!* There is a subtlety here. The word "orthonormal" could be with reference to some non-standard inner product on $\mathbb{F}^n$, whereas our work in Section 5.3 was exclusively in terms of the standard inner product.

One way to resolve this conundrum is: when we try to create the matrix A of L, we shouldn't pick *any* old basis $\mathcal{B}$ for V, but rather we should pick an *orthonormal* basis. This ensures that once we convert everything into $\mathcal{B}$-coordinates, the inner product on V will turn into the standard inner product on $\mathbb{F}^n$ (where $n = \dim V$). Indeed, recall from Example 5.4.3 that the Gram matrix of $\langle\ ,\ \rangle$ with respect to an orthonormal basis is the identity matrix, and therefore

$$\langle \vec{v}, \vec{w} \rangle = [\vec{w}]_{\mathcal{B}}^* [\vec{v}]_{\mathcal{B}}.$$

In this equation, the left-side is the inner product of the vectors $\vec{v}$ and $\vec{w}$ in V, and the right-side is the standard inner product of their $\mathcal{B}$-coordinate vectors in $\mathbb{F}^n$. Thus, the notion of orthogonality of vectors in V is equivalent to the notion of orthogonality of their $\mathcal{B}$-coordinate vectors in $\mathbb{F}^n$ with respect to the standard inner product.

This now allows us to apply all our work in Section 5.3 to the situation at hand. The upshot is:

Theorem 5.4.6 **(Spectral Theorem for Operators)**

Let $L\colon V \to V$ be a linear operator on a finite-dimensional inner product space over $\mathbb{F}$. Let $\mathcal{B}$ be an *orthonormal* basis for V and let $A = [L]_{\mathcal{B}}$. Then:

(a) If $\mathbb{F} = \mathbb{C}$, there is an orthonormal basis for V consisting of eigenvectors of L if and only if A is normal.

(b) If $\mathbb{F} = \mathbb{R}$, there is an orthonormal basis for V consisting of eigenvectors of L if and only if A is symmetric.

Proof: This follows from our preceding discussion and the corresponding spectral theorems for matrices (Theorems 5.3.8 and 5.3.6). □

We emphasize once more that the previous theorem only works if $\mathcal{B}$ is an orthonormal basis for V. See the following example.

Example 5.4.7 Endow $V = \mathcal{P}_1(\mathbb{R})$ with the inner product

$$\langle p, q\rangle = p(0)q(0) + p(1)q(1).$$

(Check that this is indeed an inner product using what we learned in the previous section!) Consider the linear operator $L\colon V \to V$ defined by

$$L(a + bx) = (a + b) + (a + b)x.$$

If we let $\mathcal{S} = \{1, x\}$ be the standard basis for $\mathcal{P}_1(\mathbb{R})$, then the corresponding matrix of L is

$$A = [L]_{\mathcal{S}} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Notice that A is symmetric. We'll leave it to you to check that the eigenvalues of A are 0 and 2, with eigenspaces

$$E_0 = \operatorname{Span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\} \quad \text{and} \quad E_2 = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}.$$

Converting back to polynomials, we see that $-1 + x$ and $1 + x$ span the two corresponding eigenspaces of L. So there is no chance of finding orthogonal eigenvectors for L, since $-1+x$ and $1 + x$ are not orthogonal! Indeed,

$$\langle -1 + x, 1 + x\rangle = (-1)(1) + (-1 + 1)(1 + 1) = -1.$$

This does **not** contradict the spectral theorem. Although the matrix A is symmetric, the problem here is that the standard basis $\mathcal{S}$ is not orthonormal with respect to $\langle\ ,\ \rangle$. Let's find an orthonormal basis and see what happens. Applying the Gram–Schmidt procedure to $\mathcal{S}$, we arrive at the orthonormal basis $\mathcal{B} = \left\{\frac{1}{\sqrt{2}}, \sqrt{2}\left(x - \frac{1}{2}\right)\right\}$. The matrix of L with respect to $\mathcal{B}$ is

$$[L]_{\mathcal{B}} = \frac{1}{2}\begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix}$$

which is **not** symmetric, so L cannot be orthogonally diagonalized, according to the spectral theorem, and as we discovered above.

We close this chapter with a remark that gives a different perspective on what we did in this section.

REMARK

Our approach to the problem of finding an orthonormal basis for an operator on an arbitrary inner product space is not the only one. We decided to tackle it using matrices to keep everything as concrete as possible. It is possible to attack the problem head on, without choosing any bases, and without reducing to the case of matrices.

The key result is that every operator $L\colon V \to V$ on a finite-dimensional inner product space gives rise to another operator $L^*\colon V \to V$, called its *adjoint*. These two operators are linked by the following fundamental property:

$$\langle L(\vec{v}), \vec{u}\rangle = \langle \vec{v}, L^*(\vec{u})\rangle \qquad \text{for all } \vec{v}, \vec{u} \in V.$$

(Compare Proposition 5.1.10.)

Starting from this innocent seeming identity, it is possible to develop the theory of the adjoint operator all the way to proving the spectral theorem for operators. From this we can then deduce the spectral theorems for matrices. This is the reverse of what we've done!

The connection between the two approaches is the following. If we choose an orthonormal basis $\mathcal{B}$ for V, then the $\mathcal{B}$-matrices $[L]_{\mathcal{B}}$ and $[L^*]_{\mathcal{B}}$ turn out to be conjugate-transposes of one another. That is, $[L]_{\mathcal{B}}^* = [L^*]_{\mathcal{B}}$. (This is not true if $\mathcal{B}$ is not orthonormal.) So our two notions of "adjoint" are thus linked.

A natural question is: why do this? Why be unsatisfied with working with matrices as we had done? Here are two reasons.

1. Our approach to the spectral theorem for operators involved *choosing* an orthonormal basis for the underlying inner product space only to later discard this basis in favour of a diagonalizing one. This seems odd. More-so because there is no obvious choice to be made at the outset, besides picking a random basis and applying Gram–Schmidt to it.

2. The matrix approach is restricted to finite-dimensional inner product spaces. The theory of orthonormal diagonalization has important consequences for operators on infinite-dimensional inner product spaces (e.g. it is of great use in quantum mechanics). As such, an approach that is not tied down to matrices is very desirable.

5.5 Application: Classifying Quadratic Forms

There are many situations where you might find yourself interested in maximizing or minimizing a certain quantity. A physicist would want to determine when a mass moving down a hill reaches a stable equilibrium, which will be the case when the potential energy due to gravity is minimized. A statistician carrying out an experiment will want to minimize the error between real-world observations and the predictions of their model. A student of linear algebra will want to solve a system of equations $A\vec{x} = \vec{b}$, which will amount to minimizing $\|A\vec{x} - \vec{b}\|$ or, equivalently, minimizing $\|A\vec{x} - \vec{b}\|^2$. (We've already considered this last problem when we discussed the method of least squares in Section 4.7.)

In the simplest of these situations, you will be faced with the task of minimizing some kind of *quadratic* function. For instance, to solve the system

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \vec{x} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

by means of solving a minimization problem as described above, you will want to minimize the quantity

$$\begin{aligned} f(\vec{x}) &= \left\| \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \vec{x} - \begin{bmatrix} 5 \\ 11 \end{bmatrix} \right\|^2 \\ &= (x_1 + 2x_2 - 5)^2 + (3x_1 + 4x_2 - 11)^2 \\ &= 10x_1^2 + 28x_1x_2 + 20x_2^2 - 76x_1 - 108x_2 + 146. \end{aligned}$$

Calculus teaches us that the local maximum and minimum values of a sufficiently differentiable function $f(\vec{x})$ (where $\vec{x} \in \mathbb{R}^n$) occur at the *critical points* of f, i.e. the points where all the partial derivatives of f vanish:

$$\frac{\partial f}{\partial x_i}(\vec{x}) = 0 \qquad \text{for all } i = 1, \ldots, n.$$

To determine whether a critical point gives a local maximum or minimum value of f (or neither), one employs a type of *second derivative test*. The idea is that near a critical point $\vec{a}$, f can be approximated by its second degree Taylor polynomial

$$f(\vec{x}) \approx f(\vec{a}) + \sum_{i=1}^{n} \underbrace{\frac{\partial f}{\partial x_i}(\vec{a})}_{=0}(x_i - a_i) + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a})(x_i - a_i)(x_j - a_j). \tag{5.1}$$

The difference $f(\vec{x}) - f(\vec{a})$ will therefore be approximated by the quadratic quantity

$$\frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a})(x_i - a_i)(x_j - a_j). \tag{5.2}$$

For instance, if (5.2) is positive for all $\vec{x} \approx \vec{a}$, then we will have $f(\vec{x}) > f(\vec{a})$ for all $\vec{x} \approx \vec{a}$, and so $f(\vec{a})$ will be a local minimum value. Similarly, if (5.2) is negative for all $\vec{x} \approx \vec{a}$, $f(\vec{a})$ will be a local maximum. Thus we find ourselves interested in determining the sign of (5.2).

To make the previous analysis rigorous, we need to carry out a careful examination of the approximation (5.1). However, if $f(\vec{x})$ is itself already a *quadratic* polynomial, we can side-step this issue, since f will be *equal* to its degree-2 Taylor polynomial:

$$f(\vec{x}) = f(\vec{a}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a})(x_i - a_i) + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a})(x_i - a_i)(x_j - a_j). \tag{5.3}$$

In fact, what is going on here is a change of basis in the space of degree ≤ 2 polynomials in $x_1, \ldots, x_n$ from

$$\{1, x_1^2, \ldots, x_n^2, x_1x_2, \ldots, x_{n-1}x_n\}$$

to

$$\{1, (x_1 - a_1)^2, \ldots, (x_n - a_n)^2, (x_1 - a_1)(x_2 - a_2), \ldots, (x_{n-1} - a_{n-1})(x_n - a_n)\}.$$

Example 5.5.1 If $f(x_1, x_2) = 10x_1^2 + 28x_1x_2 + 20x_2^2 - 76x_1 - 108x_2 + 146$ then f has the unique critical point $\vec{a} = (1, 2)$, and if we try to re-write f in terms of powers of $(x_1 - 1)$ and $(x_2 - 2)$, we would find that

$$f(x_1, x_2) = 10(x_1 - 1)^2 + 28(x_1 - 1)(x_2 - 2) + 20(x_2 - 2)^2.$$

We'll leave it to you to check that this is true, and to verify that the above expression is identical to

$$f(x_1, x_2) = f(1,2) + \sum_{i=1}^{2} \frac{\partial f}{\partial x_i}(1,2)(x_i - a_i) + \frac{1}{2}\sum_{i,j=1}^{2} \frac{\partial^2 f}{\partial x_i \partial x_j}(1,2)(x_i - a_i)(x_j - a_j).$$

Returning to (5.2), if we let

$$u_i = x_i - a_i \quad \text{and} \quad a_{ij} = \frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a})$$

then the expression takes the simpler form

$$\sum_{i,j=1}^{n} a_{ij} u_i u_j.$$

Definition 5.5.2
Quadratic Form

A (real) **quadratic form** in the variables $\vec{u} = (u_1, \ldots, u_n)$ is a polynomial of the form

$$Q(\vec{u}) = \sum_{i,j=1}^{n} a_{ij} u_i u_j, \qquad \text{where } a_{ij} \in \mathbb{R}.$$

Thus, a quadratic form is effectively a quadratic polynomial that does not have any linear or constant terms. We are interested in determining the *sign* of a quadratic form. Observe that $Q(\vec{0}) = 0$.

Definition 5.5.3
Positive Definite, Negative Definite, Semi-Definite, Indefinite Quadratic Form

A quadratic form $Q(\vec{u})$ is said to be

- **positive definite** if $Q(\vec{u}) > 0$ for all non-zero $\vec{u} \in \mathbb{R}^n$;
- **positive semi-definite** if $Q(\vec{u}) \geq 0$ for all $\vec{u} \in \mathbb{R}^n$;
- **negative definite** if $Q(\vec{u}) < 0$ for all non-zero $\vec{u} \in \mathbb{R}^n$;
- **negative semi-definite** if $Q(\vec{u}) \leq 0$ for all $\vec{u} \in \mathbb{R}^n$;
- **indefinite** if there exist $\vec{u}, \vec{v} \in \mathbb{R}^n$ such that $Q(\vec{u}) > 0$ and $Q(\vec{v}) < 0$.

Notice that every positive (resp. negative) definite quadratic form is also positive (negative) semi-definite.

Example 5.5.4

The quadratic form $Q_1(u_1, u_2) = u_1^2 + u_2^2$ is positive definite.

The quadratic form $Q_2(u_1, u_2) = u_1^2$ is positive semi-definite but not positive definite.

The quadratic form $Q_3(u_1, u_2) = -3u_1^2 - 4u_2^2$ is negative definite.

The quadratic form $Q_4(u_1, u_2) = -4u_1^2$ is negative semi-definite.

The quadratic form $Q_5(u_1, u_2) = 3u_1^2 - 2u_2^2$ is indefinite.

The verification of the above is entirely trivial. But how do we classify something more complicated like $Q_6(u_1, u_2) = 10u_1^2 + 28u_1u_2 + 20u_2^2$? (This is the quadratic form that appeared in the previous example.)

Our main tool in being able to classify a quadratic form into one of the above classes is its associated matrix. Consider an arbitrary two-variable quadratic form

$$Q(u_1, u_2) = \sum_{i,j=1}^{2} a_{ij}u_iu_j = a_{11}u_1^2 + a_{12}u_1u_2 + a_{21}u_2u_1 + a_{22}u_2^2.$$

We can re-write this as

$$Q(u_1, u_2) = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

or more simply as

$$Q(\vec{u}) = \vec{u}^T \begin{bmatrix} a_{11} & \frac{a_{12}+a_{21}}{2} \\ \frac{a_{12}+a_{21}}{2} & a_{22} \end{bmatrix} \vec{u}.$$

(This type of expression should ring some bells. Compare it to our construction of the Gram matrix of an inner product in Section 5.4.) The same kind of construction is possible for n-variable quadratic forms:

Proposition 5.5.5

Let $Q(\vec{u}) = \sum_{i,j=1}^{n} a_{ij}u_iu_j$ be a quadratic form. If we let A be the $n \times n$ matrix whose (i, j)th entry is $\frac{a_{ij} + a_{ji}}{2}$, then

$$Q(\vec{u}) = \vec{u}^T A\vec{u}.$$

EXERCISE

Prove Proposition 5.5.5.

Definition 5.5.6

Matrix Associated to a Quadratic Form

The matrix $A \in M_{n\times n}(\mathbb{R})$ constructed in Proposition 5.5.5 is called the **matrix associated to the quadratic form** $Q(\vec{u})$.

Example 5.5.7

The matrix associated to the quadratic form $Q(u_1, u_2) = 10u_1^2 + 28u_1u_2 + 20u_2^2$ is

$$A = \begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix}.$$

Notice that the diagonal entries are the coefficients of the pure square terms u_1^2 and u_2^2 while the off-diagonal entries are *one-half* of the coefficient of the mixed term u_1u_2.

Example 5.5.8

The matrix associated to the quadratic form $Q(u_1, u_2, u_3) = 3u_1^2 + u_1u_2 - 2u_2u_3 + 3u_3^2$ is

$$A = \begin{bmatrix} 3 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & -1 \\ 0 & -1 & 3 \end{bmatrix}.$$

Again, notice that the diagonal entries are the coefficients of the pure square terms, while the off-diagonal entries are one-half the coefficients of the mixed terms.

The key thing to glean from the previous two examples is that the matrices we got were symmetric. This is true in general.

Proposition 5.5.9

Let $A \in M_{n\times n}(\mathbb{R})$ be the matrix associated to the quadratic form $Q(\vec{u})$. Then A is symmetric.

EXERCISE

Prove Proposition 5.5.9.

REMARK

We can reverse the sequence of ideas presented above. Starting from a symmetric matrix $A \in M_{n\times n}(\mathbb{R})$, we can create a quadratic form $Q(\vec{u}) = \vec{u}^T A\vec{u}$. Propositions 5.5.5 and 5.5.9 guarantee that in this way we are able to produce all quadratic forms.

Thus, quadratic forms in n-variables and $n \times n$ symmetric matrices are essentially one and the same. Can you formulate this as some kind of isomorphism between two vector spaces?

Given this, we can now appeal to the spectral theorem for symmetric matrices (Theorem 5.3.6). The upshot is the following result, which says that the sign of $Q(\vec{u})$ is determined by the signs of the eigenvalues of A.

Theorem 5.5.10

(Classification of Quadratic Forms)

Let $Q(\vec{u})$ be a quadratic form with associated matrix A. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of A. Then:

1. $Q(\vec{u})$ is positive definite if and only if $\lambda_i > 0$ for all $i \in \{1, \dots, n\}$.
2. $Q(\vec{u})$ is positive semi-definite if and only if $\lambda_i \geq 0$ for all $i \in \{1, \dots, n\}$.
3. $Q(\vec{u})$ is negative definite if and only if $\lambda_i < 0$ for all $i \in \{1, \dots, n\}$.
4. $Q(\vec{u})$ is negative semi-definite if and only if $\lambda_i \leq 0$ for all $i \in \{1, \dots, n\}$.
5. $Q(\vec{u})$ is indefinite if and only if $\lambda_i > 0$ and $\lambda_j < 0$ for some $i, j \in \{1, \dots, n\}$.

Proof: By the spectral theorem for symmetric matrices (Theorem 5.3.6), we can orthogonally diagonalize A. That is, there exists an orthogonal matrix $P \in M_{n\times n}(\mathbb{R})$ such that $A = PDP^T$, where $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. Without loss of generality, let's label the eigenvalues by descending order according to size: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. (Note that the eigenvalues are all real, by the spectral theorem, so this makes sense.) Then

$$Q(\vec{u}) = \vec{u}^T A \vec{u} = \vec{u}^T PDP^T \vec{u} = (P^T\vec{u})^T D (P^T\vec{u}).$$

If we let $\vec{y} = P^T\vec{u} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ (equivalently, $\vec{u} = P\vec{y}$), then we can write

$$Q(\vec{u}) = \vec{y}^T \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \vec{y} = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2.$$

The sign of Q is now very easy to determine. For instance, if all the λ_i are positive, then $Q(\vec{u}) > 0$ for all $\vec{u} \neq \vec{0}$. This proves (a). The proofs of (b), (c) and (d) are similar. For (e), consider $\vec{u} = P\vec{y}$, wher $\vec{y} = \vec{e}_i$ and $\vec{y} = \vec{e}_j$ (where $\vec{e}_k$ is the kth standard basis vector for $\mathbb{R}^n$). Then $Q(\vec{u}) = \lambda_i > 0$ and $Q(\vec{u}) = \lambda_j < 0$, respectively, proving that Q is indefinite. □

The take-away from the above proof is that when we diagonalize A, the quadratic form becomes simple: all the mixed terms disappear, and we are left only with pure terms. (Essentially, we *completed the squares*.) This is generally what diagonalization does. It removes unnecessary complications that are present because we are, in a sense, working with a less than optimal point of view.

Example 5.5.11

The matrix associated to the quadratic form $Q(u_1, u_2) = 10u_1^2 + 28u_1u_2 + 20u_2^2$ is

$$A = \begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix}.$$

Its eigenvalues are

$$\lambda_1 = 15 + \sqrt{221} \quad \text{and} \quad \lambda_2 = 15 - \sqrt{221}.$$

Since these are positive, we conclude that $Q(\vec{u})$ is positive definite.

Example 5.5.12

The matrix associated to the quadratic form $Q(u_1, u_2, u_3) = 3u_1^2 + u_1u_2 - 2u_2u_3 + 3u_3^2$ is

$$A = \begin{bmatrix} 3 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & -1 \\ 0 & -1 & 3 \end{bmatrix}.$$

Its eigenvalues are

$$\lambda_1 = \frac{3+\sqrt{14}}{2}, \quad \lambda_2 = 3 \quad \text{and} \quad \lambda_3 = \frac{3-\sqrt{14}}{2}.$$

Since $\lambda_1 > 0$ and $\lambda_3 < 0$, it follows that $Q(\vec{u})$ is indefinite.

Let's return now to our motivating problem of optimizing a quadratic function $f(\vec{x})$. We had reduced the issue to determining the sign of the quadratic form

$$Q(\vec{u}) = \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a})u_i u_j,$$

where $\vec{a}$ is a critical point of the function. The matrix associated to Q is $A = \frac{1}{2}H$, with

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(\vec{a}) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(\vec{a}) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(\vec{a}) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(\vec{a}) & \frac{\partial^2 f}{\partial x_2^2}(\vec{a}) & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n}(\vec{a}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(\vec{a}) & \frac{\partial^2 f}{\partial x_n \partial x_2}(\vec{a}) & \cdots & \frac{\partial^2 f}{\partial x_n^2}(\vec{a}) \end{bmatrix},$$

where we've implicitly used the fact that for a sufficiently differentiable function f, its mixed second-order partial derivatives are equal:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

The matrix H above is called the **Hessian matrix** of f.It plays a role analogous to the second derivative of a single-variable function. Our discussion above tells us that the nature of the critical point $\vec{a}$ is tied to the sign of Q hence to the eigenvalues of H. In particular, $f(\vec{a})$ will be a local maximum value of f if Q is negative definite, and will be a local minimum if Q is positive definite. This is part of the so-called *second derivative test for multivariable functions*. To learn more, take a course in multivariable calculus!

Example 5.5.13 Let's revisit our function

$$f(x_1, x_2) = \left\| \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \vec{x} - \begin{bmatrix} 5 \\ 11 \end{bmatrix} \right\|^2 = 10x_1^2 + 28x_1x_2 + 20x_2^2 - 76x_1 - 108x_2 + 146.$$

Our work in Example 5.5.1 shows that the Hessian matrix of f at the critical point $\vec{a} = (1,2)$ is

$$H = 2A = \begin{bmatrix} 20 & 28 \\ 28 & 40 \end{bmatrix},$$

where A is the matrix associated to the quadratic form $Q(u_1, u_2) = 20u_1^2 + 28u_1u_2 + 40u_2^2$. In Example 5.5.11, we showed that Q is positive-definite. Thus, $f(1,2) = 0$ is a local minimum value of f. In fact, since $f(x_1, x_2) \geq 0$ for all (x_1, x_2), it must be the case that this is a *global* minimum value.

Of course, this is to be expected, since $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a solution to the system

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \vec{x} = \begin{bmatrix} 5 \\ 11 \end{bmatrix},$$

hence definitely minimizes $f(x_1, x_2)$! In fact, we know that $(x_1, x_2) = (1,2)$ gives us the unique global minimum value of f. But it's good to know that our methods above allow us to reach this same conclusion. Their true power emerges once we consider more complicated (in particular, non-quadratic) functions f.

Chapter 6

The Singular Value Decomposition

6.1 Singular Values and Singular Vectors

In Chapter 3 we studied the problem of finding a simple matrix representation of a given linear operator $L\colon V \to V$ on a finite-dimensional vector space V. We learned that, if certain conditions are satisfied by L, we're able to find a basis $\mathcal{D}$ for V consisting of eigenvectors of L such that $[L]_{\mathcal{D}}$ is a diagonal matrix. Unfortunately, since the needed conditions are not always met, many linear operators are left without simple matrix representations.

In this chapter we aim to rectify this situation. What made diagonalization difficult to attain is our insistence on using the same basis $\mathcal{D}$ for both the domain and codomain of $L\colon V \to V$. Let's instead ask for two (possibly different) bases $\mathcal{B}$ and $\mathcal{C}$ of V so that ${}_{\mathcal{C}}[L]_{\mathcal{B}}$ is diagonal. Are we always able to find these? The answer is *yes*! Actually, this is quite easy to do—but what is more interesting is that, if V is an inner product space, we can arrange for $\mathcal{B}$ and $\mathcal{C}$ to be orthonormal bases. Additionally, all this can be done not just for operators, but for linear maps $L\colon V \to W$ as well! (Of course, in this case ${}_{\mathcal{C}}[L]_{\mathcal{B}}$ won't be a square matrix, so we'll need to redefine what we mean by "diagonal matrix"—but you can probably guess what the definition is going to be.)

Although this seems like it would be only of theoretical interest, this is very far from the truth. The results of this chapter in fact have an abundance of practical, real-world applications—see Section 6.4.

We will begin by seeing how all this works for a matrix $A \in M_{m\times n}(\mathbb{F})$, which we think of as giving a linear map $\mathbb{F}^n \to \mathbb{F}^m$. We give $\mathbb{F}^n$ and $\mathbb{F}^m$ their standard inner product. The key to our approach to diagonalization of square matrices (and linear operators) was the idea of eigenvectors and eigenvalues. Our starting point here will be to find a suitable analogue for non-square matrices; these are the so-called *singular vectors* and *singular values* of A, to be introduced momentarily. First we need a preliminary lemma.

Lemma 6.1.1 Let $A \in M_{m\times n}(\mathbb{F})$. Then A^*A is an $n \times n$ Hermitian matrix and its eigenvalues are non-negative real numbers.

Proof: It's clear that A^*A is $n \times n$ and Hermitian. Let λ be an eigenvalue of A^*A, say with eigenvector $\vec{x}$. Then $A^*A\vec{x} = \lambda\vec{x}$, and therefore

$$\lambda\|\vec{x}\|^2 = \lambda\langle\vec{x},\vec{x}\rangle = \langle\lambda\vec{x},\vec{x}\rangle = \langle A^*A\vec{x},\vec{x}\rangle = \langle A\vec{x},A\vec{x}\rangle = \|A\vec{x}\|^2.$$

Since $\vec{x}$ is an eigenvector, $\|\vec{x}\| \neq 0$, and consequently $\lambda = \dfrac{\|A\vec{x}\|^2}{\|\vec{x}\|^2} \geq 0$. □

Definition 6.1.2

Singular Values, Singular Vectors

Let $A \in M_{m\times n}(\mathbb{F})$. The **singular values** of A are the non-negative square-roots $\sigma_i = \sqrt{\lambda_i}$ of the eigenvalues λ_i of A^*A.

The corresponding eigenvectors of A^*A are called the **singular vectors** of A.

REMARK

The name *singular value* originates in the theory of integral equations, and was coined by Emile Picard for a value that is of special (or *singular*) interest. It has nothing to do with our modern mathematical usage of the word "singular."

By convention, the singular values are always ordered in descending order:

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0.$$

At this stage it's not at all clear what singular values and vectors have to do with our motivating problem. The connection is explained in the discussion preceding Theorem 6.2.1 in the next section. For now, however, let's see some examples.

Example 6.1.3

Let $A = \begin{bmatrix} 1 & -4 \\ -2 & 2 \\ 2 & 4 \end{bmatrix}$. Then

$$A^*A = A^TA = \begin{bmatrix} 1 & -2 & 2 \\ 4 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ -2 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 36 \end{bmatrix}.$$

(Notice that this matrix is symmetric, as expected.)

The eigenvalues of A^*A are $\lambda_1 = 36$ and $\lambda_2 = 9$. (Notice that they are non-negative, as expected.) Thus the singular values of A are $\sigma_1 = \sqrt{36} = 6$ and $\sigma_2 = \sqrt{9} = 3$.

The vectors $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ are corresponding singular vectors, since they are eigenvectors for A^*A with eigenvalues $\lambda_1 = 36$ and $\lambda_2 = 9$, respectively.

Example 6.1.4

Let $A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$. Then

$$A^*A = A^TA = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & 1 \end{bmatrix}.$$

You can check that the eigenvalues of A^*A are $\lambda_1 = 3$, $\lambda_2 = 2$ and $\lambda_3 = 0$. Thus the singular values of A are $\sigma_1 = \sqrt{3}$, $\sigma_2 = \sqrt{2}$ and $\sigma_3 = 0$. Corresponding singular vectors are $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, respectively.

Example 6.1.5 Let $A = \begin{bmatrix} 0 & 2 \\ -3 & 0 \end{bmatrix}$. Then

$$A^*A = A^TA = \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}.$$

The eigenvalues of A^*A are $\lambda_1 = 9$ and $\lambda_2 = 4$. Thus the singular values of A are $\sigma_1 = 3$ and $\sigma_2 = 2$, and $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are corresponding singular vectors.

Example 6.1.6 Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \\ i & 2i & -i \end{bmatrix}$. Then

$$A^*A = \begin{bmatrix} 1 & 2 & -1 & -i \\ 2 & 4 & -2 & -2i \\ -1 & -2 & 1 & i \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \\ i & 2i & -i \end{bmatrix} = \begin{bmatrix} 7 & 14 & -7 \\ 14 & 28 & -14 \\ -7 & -14 & 7 \end{bmatrix}.$$

The eigenvalues of A^*A are $\lambda_1 = 42$ and $\lambda_2 = \lambda_3 = 0$. For eigenvectors, we can take $\vec{v}_1 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, respectively. Here we decided to pick two orthogonal eigenvectors $\vec{v}_2$ and $\vec{v}_3$ for the repeated eigenvalue 0. We are able to do this because A^*A is symmetric, hence orthogonally diagonalizable by the spectral theorem.

The singular values of A are $\sigma_1 = \sqrt{42}$ and $\sigma_2 = \sigma_3 = 0$, with corresponding singular vectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ as above.

EXERCISE

Let $A \in M_{n\times n}(\mathbb{F})$ be a Hermitian matrix. What are the singular values of A? What can you say about the corresponding singular vectors?

There are two things we can notice from the previous examples:

1. The singular vectors of A corresponding to different singular values are orthogonal.
2. The rank of A is equal to the number of non-zero singular values of A.

These resemble two facts that we know about eigenvalues and eigenvectors of a square matrix A. Namely, eigenvectors corresponding to distinct eigenvalues are linearly independent (Proposition 3.2.10), and the nullity of A is equal to the geometric multiplicity of 0 as an eigenvalue; hence if A is diagonalizable, nullity(A) is equal to the algebraic multiplicity of 0, and so rank(A) is the sum of the algebraic multiplicities of the non-zero eigenvalues which we can interpret as being the number of non-zero eigenvalues, each counted according to multiplicity.

Let's now prove that our two observations above are in fact always true.

Proposition 6.1.7 Let $A \in M_{m\times n}(\mathbb{F})$, and let $\vec{x}, \vec{y} \in \mathbb{F}^n$ be singular vectors of A corresponding to the singular values σ_1 and σ_2. If $\sigma_1 \neq \sigma_2$, then $\vec{x}$ and $\vec{y}$ are orthogonal.

Proof: Let $B = A^*A$ and let $\lambda_i = \sigma_i^2$. Notice that $\lambda_1 \neq \lambda_2$ since otherwise we'd have $\sigma_1 = \pm\sigma_2$ hence $\sigma_1 = \sigma_2$ since both are non-negative. Thus, $\vec{x}$ and $\vec{y}$ are eigenvectors of B with distinct eigenvalues λ_1 and λ_2, respectively. Since B is Hermitian, hence normal, the orthogonality of $\vec{x}$ and $\vec{y}$ follows from Proposition 5.3.9(c). □

To prove our second observation, we will need the following lemma.

Lemma 6.1.8 Let $A \in M_{m\times n}(\mathbb{F})$. Then $\mathrm{Null}(A^*A) = \mathrm{Null}(A)$.

Proof: Suppose that $\vec{x} \in \mathrm{Null}(A)$, so that $A\vec{x} = \vec{0}$. Then $A^*A\vec{x} = A^*\vec{0} = \vec{0}$, so $\vec{x} \in \mathrm{Null}(A^*A)$. Thus, $\mathrm{Null}(A) \subseteq \mathrm{Null}(A^*A)$. Conversely, suppose that $\vec{x} \in \mathrm{Null}(A^*A)$. Then

$$\|A\vec{x}\|^2 = \langle A\vec{x}, A\vec{x}\rangle = \langle \vec{x}, A^*A\vec{x}\rangle = \langle \vec{x}, \vec{0}\rangle = 0$$

so that $A\vec{x} = \vec{0}$, proving that $\vec{x} \in \mathrm{Null}(A)$ and hence that $\mathrm{Null}(A^*A) \subseteq \mathrm{Null}(A)$. This completes the proof. □

Proposition 6.1.9 Let $A \in M_{m\times n}(\mathbb{F})$. The number of non-zero singular values of A is equal to rank(A), where each repeated singular value is counted according to its multiplicity. (The *multiplicity* of a singular value σ is the algebraic multiplicity of σ^2 as an eigenvalue of A^*A.)

Proof: Since A^*A is Hermitian, hence diagonalizable by the spectral theorem, the argument preceding Proposition 6.1.7 shows that rank(A^*A) is equal to the number of non-zero eigenvalues of A^*A, which in turn is equal to the number of singular values of A, each counted according to multiplicity. (Here we're again using the fact that distinct singular values of A come from distinct eigenvalues of A^*A, and vice versa, as observed in the proof of Proposition 6.1.7.)

So the proof will be complete if we can show that rank$(A) = $ rank(A^*A). Since A is $m \times n$ and A^*A is $n \times n$, the rank-nullity theorem and Lemma 6.1.8 give

$$\mathrm{rank}(A) = n - \mathrm{nullity}(A) = n - \mathrm{nullity}(A^*A) = \mathrm{rank}(A^*A),$$

as desired. □

In particular, an $m \times n$ matrix will have at most $\min\{m, n\}$ non-zero singular values.

6.2 Singular Value Decomposition of Matrices

In this section we will show that we can always diagonalize a matrix $A \in M_{m\times n}(\mathbb{F})$ by choosing appropriate orthonormal bases for $\mathbb{F}^n$ and $\mathbb{F}^m$. If we let U and V be the matrices whose columns are these basis vectors, then this amounts to a factorization of A of the form

$$A = U\Sigma V^*,$$

where U is an $m \times m$ unitary matrix, V is an $n \times n$ unitary matrix, and where Σ is an $m \times n$ matrix whose (i, j)th entry is 0 for $i \neq j$.

The idea behind the proof is really simple: we reverse engineer what we want. That is, suppose we know that $A = U\Sigma V^*$. Then we'd also have that $A^* = V\Sigma^* U^*$ and therefore

$$A^*A = V\Sigma^*U^*U\Sigma V^* = V\Sigma^*\Sigma V^*.$$

The matrix $\Sigma^*\Sigma$ is a square diagonal matrix whose diagonal entries are non-negative real numbers. So, we've conclude that **if** $A = U\Sigma V^*$, then we'd be able to unitarily diagonalize A^*A with non-negative real eigenvalues. But since A^*A is Hermitian, we *know* that this is possible thanks to the spectral theorem (Theorem 5.3.5) and Lemma 6.1.1. So this suggests what to take for the diagonal entries of Σ and for V: namely, the positive square roots of the eigenvalues of A^*A (i.e. the singular values of A), and the same V that unitarily diagonalizes A^*A (i.e. the corresponding singular vectors)!

By applying the same analysis to $AA^* = U\Sigma\Sigma^*U^*$, we know what we must take for U. Now all that remains is to show that if we do take these U, Σ and V, then we get our desired decomposition $A = U\Sigma V^*$. The proof below will basically do this, except it will construct U directly because this makes the verification easier.

Theorem 6.2.1 **(Singular Value Decomposition of Matrices)**

Let $A \in M_{m\times n}(\mathbb{F})$ be a matrix of rank r with non-zero singular values $\sigma_1 \geq \cdots \geq \sigma_r > 0$. Then there exist unitary matrices $U \in M_{m\times m}(\mathbb{F})$ and $V \in M_{n\times n}(\mathbb{F})$ such that

$$A = U\Sigma V^*,$$

where Σ is the $m \times n$ matrix whose entries are

$$\Sigma_{ij} = \begin{cases} \sigma_i & \text{if } i = j \leq r \\ 0 & \text{otherwise.} \end{cases}$$

If $A \in M_{m\times n}(\mathbb{R})$ is real, then U and V can be chosen to be orthogonal matrices.

Proof: The $n \times n$ Hermitian matrix A^*A is unitarily diagonalizable, by the spectral theorem. So we can find an orthonormal basis $\{\vec{v}_1, \dots, \vec{v}_n\}$ for $\mathbb{F}^n$ consisting of eigenvectors of A^*A with corresponding eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ (which are all real, by the spectral theorem, so it is possible to order them like this). Then $V = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$ is a unitary matrix and

$$AV = \begin{bmatrix} A\vec{v}_1 \cdots A\vec{v}_n \end{bmatrix}.$$

According to Proposition 6.1.9, $\lambda_i = \sigma_i^2 = 0$ for all $i > r$. Thus the vectors $\vec{v}_i$ for $i > r$ are all in $\mathrm{Null}(A^*A)$, hence in $\mathrm{Null}(A)$ by Lemma 6.1.8. So

$$AV = \begin{bmatrix} A\vec{v}_1 & \cdots & A\vec{v}_r & \vec{0} & \cdots & \vec{0} \end{bmatrix}$$

Let $\vec{u}_i = \frac{1}{\sigma_i} A\vec{v}_i$ for $i \leq r$ and notice that, since σ_i is real and positive,

$$\langle \vec{u}_i, \vec{u}_j \rangle = \frac{1}{\sigma_i \sigma_j} \langle A\vec{v}_i, A\vec{v}_j \rangle = \frac{1}{\sigma_i \sigma_j} \langle \vec{v}_i, A^* A\vec{v}_j \rangle = \frac{1}{\sigma_i \sigma_j} \left\langle \vec{v}_i, \sigma_j^2 \vec{v}_j \right\rangle = \frac{\sigma_j}{\sigma_i} \langle \vec{v}_i, \vec{v}_j \rangle .$$

Thus, $\langle \vec{u}_i, \vec{u}_j \rangle$ is 0 for $i \neq j$ and is 1 otherwise, since $\{\vec{v}_1 \ldots, \vec{v}_n\}$ is an orthonormal set. It follows that $\{\vec{u}_1, \ldots, \vec{u}_r\}$ is an orthonormal set in $\mathbb{F}^m$. Extend it to an orthonormal basis $\{\vec{u}_1, \ldots, \vec{u}_m\}$. Then $U = \begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_m \end{bmatrix}$ is unitary and we have

$$\begin{aligned} AV &= \begin{bmatrix} A\vec{v}_1 & \cdots & A\vec{v}_r & \vec{0} & \cdots & \vec{0} \end{bmatrix} \\ &= \begin{bmatrix} \sigma_1 \vec{u}_1 & \cdots & \sigma_r \vec{u}_r & \vec{0} & \cdots & \vec{0} \end{bmatrix} \\ &= \begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_r & \vec{u}_{r+1} & \cdots & \vec{u}_m \end{bmatrix} \begin{bmatrix} \operatorname{diag}(\sigma_1, \ldots, \sigma_r) & 0_{r \times (n-r)} \\ 0_{(m-r) \times r} & 0_{(n-r) \times (m-r)} \end{bmatrix} \\ &= \Sigma U. \end{aligned}$$

Multiplying both sides by V^* on the right, we get our desired decomposition $A = U\Sigma V^*$. □

EXERCISE

By examining the proof of Theorem 6.2.1, show that if $A \in M_{m \times n}(\mathbb{R})$ is real, then we can choose U and V to be orthogonal.

Definition 6.2.2
Singular Value Decomposition, SVD

A decomposition $A = U\Sigma V^*$ of the type occurring in Theorem 6.2.1 is called a **singular value decomposition (SVD)** of A.

Although the entries of Σ are uniquely determined by A (they are its singular values), there is generally quite a bit of freedom in choosing U and V. For instance, if $A = I_n$ is the $n \times n$ identity matrix, then $A = U I_n U^*$ will be an SVD of A for *any* unitary $n \times n$ unitary matrix A. Thus, an SVD is not unique.

If we let $D = \operatorname{diag}(\sigma_1, \ldots, \sigma_r) \in M_{r \times r}(\mathbb{F})$, we see that the matrix Σ in an SVD of A takes one of the following shapes, depending on $r = \operatorname{rank}(A)$, m and n:

- If $r < \min\{m, n\}$, then $\Sigma = \begin{bmatrix} D & 0_{r \times (n-r)} \\ 0_{(m-r) \times r} & 0_{(n-r) \times (m-r)} \end{bmatrix}$.
- If $r = m < n$, then $\Sigma = \begin{bmatrix} D & 0_{r \times (n-r)} \end{bmatrix}$.
- If $r = n < m$, then $\Sigma = \begin{bmatrix} D \\ 0_{(m-r) \times r} \end{bmatrix}$.
- If $r = m = n$, then $\Sigma = D$.

We will illustrate each of these scenarios below.

First, let's note that the proof of Theorem 6.2.1 describes an algorithm for constructing an SVD for a given $m \times n$ matrix A.

ALGORITHM (Finding an SVD for a Matrix)

Let $A \in M_{m\times n}(\mathbb{F})$ be a matrix of rank r. To find an SVD for A:

1. Find the eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ and a corresponding set of **orthonormal** eigenvectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$ for A^*A. (See the Algorithm on page 118.)
2. Set $\sigma_i = \sqrt{\lambda_i}$ for $i \leq r$.
3. Set $\vec{u}_i = \frac{1}{\sigma_i} A\vec{v}_i$ for $i \leq r$. If $r < m$, extend $\{\vec{u}_1, \ldots, \vec{u}_r\}$ to an orthonormal basis $\{\vec{u}_1, \ldots, \vec{u}_r, \vec{u}_{r+1}, \ldots, \vec{u}_m\}$ of $\mathbb{F}^m$.
4. Set $V = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$ and $U = \begin{bmatrix} \vec{u}_1 & \ldots & \vec{u}_m \end{bmatrix}$, and let Σ be the $m \times n$ matrix whose entries are
$$\Sigma_{ij} = \begin{cases} \sigma_i & \text{if } i = j \leq r \\ 0 & \text{otherwise.} \end{cases}$$

Then U and V are unitary square matrices (orthogonal matrices if A is real) and

$$A = U\Sigma V^*.$$

Notice that in the above algorithm we will have $A\vec{v}_i = \sigma_i \vec{u}_i$ for all $i \leq r$ (and even for $r < i \leq n$, since in that case $\sigma_i = 0$ and $A\vec{v}_i = \vec{0}$ by Lemma 6.1.8), which should remind you of eigenvectors and eigenvalues. See Section 6.2.1 for a geometric interpretation.

To facilitate Step 3 of the algorithm, let's observe that the vectors $\vec{u}_1, \ldots, \vec{u}_r$ form a set of $r = \text{rank}(A)$ linearly independent vectors in $\text{Col}(A)$, thus they must be an orthonormal basis for $\text{Col}(A)$. It follows that $\{\vec{u}_{r+1}, \ldots, \vec{u}_m\}$ must be an orthonormal basis for $\text{Col}(A)^\perp$. Here is a handy result, which tells us that one way of finding the vectors $\{\vec{u}_{r+1}, \ldots, \vec{u}_m\}$ is by taking them to be an orthonormal basis for $\text{Null}(A^*)$.

Proposition 6.2.3 Let $A \in M_{m\times n}(\mathbb{F})$. Then $\text{Col}(A)^\perp = \text{Null}(A^*)$.

Proof: The vectors in $\text{Col}(A)$ are those of the form $A\vec{x}$, where $\vec{x} \in \mathbb{F}^n$ is arbitrary. So a vector $\vec{y} \in \mathbb{F}^m$ will be in $\text{Col}(A)^\perp$ if and only if

$$0 = \langle A\vec{x}, \vec{y}\rangle = \langle \vec{x}, A^*\vec{y}\rangle \qquad \text{for all } \vec{x} \in \mathbb{F}^n,$$

which is the case if and only if $A^*\vec{y} \in (\mathbb{F}^n)^\perp = \{\vec{0}\}$, i.e., if and only if $\vec{y} \in \text{Null}(A^*)$, proving the proposition. □

Let's now compute SVDs for the matrices in Examples 6.1.3—6.1.6.

Example 6.2.4 Let $A = \begin{bmatrix} 1 & -4 \\ -2 & 2 \\ 2 & 4 \end{bmatrix}$. The singular values of A were found to be $\sigma_1 = 6$ and $\sigma_2 = 3$, with corresponding singular vectors $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (which are unit vectors). Now let

$$\vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1 = \frac{1}{3}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \vec{u}_2 = \frac{1}{\sigma_2} A\vec{v}_2 = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}.$$

Notice that $\{\vec{u}_1, \vec{u}_2\}$ is an orthonormal set in $\mathbb{F}^3$ (as guaranteed by the proof of Theorem 6.2.1). We must extend it to an orthonormal basis for $\mathbb{F}^3$. This can be achieved in a variety of ways. For instance, we can take the cross product of $\vec{u}_1$ and $\vec{u}_2$ or we can find an orthonormal basis for $\text{Null}(A^*)$. In any case, we obtain $\vec{u}_3 = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

Finally, set

$$V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -2 & 1 & 2 \\ 1 & -2 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

and notice that these matrices are orthogonal. The resulting SVD of A is:

$$A = U \begin{bmatrix} 6 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} V^T.$$

Example 6.2.5 Let $A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$. The singular values of A were found to be $\sigma_1 = \sqrt{3}$, $\sigma_2 = \sqrt{2}$ and $\sigma_3 = 0$, with corresponding singular vectors $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, respectively. Let $\vec{v}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_3 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, so that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is orthonormal. Notice that $r = \text{rank}(A) = 2$, since there are only two non-zero singular values. Next, let

$$\vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{u}_2 = \frac{1}{\sigma_2} A\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Then $\{\vec{u}_1, \vec{u}_2\}$ is already a basis for $\mathbb{F}^2$ and we can construct the orthogonal matrices

$$V = \frac{1}{\sqrt{6}}\begin{bmatrix} -\sqrt{2} & \sqrt{3} & 1 \\ \sqrt{2} & \sqrt{3} & -1 \\ \sqrt{2} & 0 & 2 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The resulting SVD of A is

$$A = U \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix} V^T.$$

Example 6.2.6 Let $A = \begin{bmatrix} 0 & 2 \\ -3 & 0 \end{bmatrix}$. The singular values of A were found to be $\sigma_1 = 3$ and $\sigma_2 = 2$, with corresponding singular vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, respectively. These are unit vectors, so we may set

$$\vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \vec{u}_2 = \frac{1}{\sigma_2} A\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Thus, in this case our orthogonal matrices are

$$V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and the resulting SVD of A is

$$A = U \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} V^T.$$

Interestingly, while A is **not** diagonalizable over $\mathbb{R}$, it is diagonalizable over $\mathbb{C}$ (with distinct eigenvalues $\pm\sqrt{-6}$). However, A is not unitarily diagonalizable since it is not normal, as you can check.

Example 6.2.7 Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \\ i & 2i & -i \end{bmatrix}$. The singular values of A were found to be $\sigma_1 = \sqrt{42}$ and $\sigma_2 = \sigma_3 = 0$ with corresponding orthonormal eigenvectors $\vec{v}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{v}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, respectively. Here there is only one non-zero singular value, and correspondingly we set

$$\vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1 = \frac{1}{\sqrt{7}} \begin{bmatrix} -1 \\ -2 \\ 1 \\ -i \end{bmatrix}.$$

Now we must extend $\{\vec{u}_1\}$ to an orthonormal basis for $\mathbb{C}^4$. We can do so by applying the Gram–Schmidt process to $\{\vec{u}_1, \vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4\}$, where the $\vec{e}_i$ are the standard basis vectors for $\mathbb{C}^4$, or by finding an orthonormal basis for $\text{Null}(A^*)$. In either case it's going to be a tedious computation! We'll spare you the boring details and just give the end result:

$$\vec{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} i \\ 0 \\ 0 \\ 1 \end{bmatrix}, \ \vec{u}_4 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 0 \\ 2 \\ i \end{bmatrix} \quad \text{and} \quad \vec{u}_3 = \frac{1}{\sqrt{21}} \begin{bmatrix} -2 \\ 3 \\ 2 \\ -2i \end{bmatrix}.$$

We thus have our unitary matrices $V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$ and $U = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \vec{u}_4 \end{bmatrix}$ with corresponding SVD

$$A = U \begin{bmatrix} \sqrt{42} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^*.$$

EXERCISE

Let $A \in M_{n\times n}(\mathbb{F})$ be a Hermitian matrix. How does an SVD $A = U\Sigma V^*$ compare to a unitary diagonalization $A = WDW^*$?

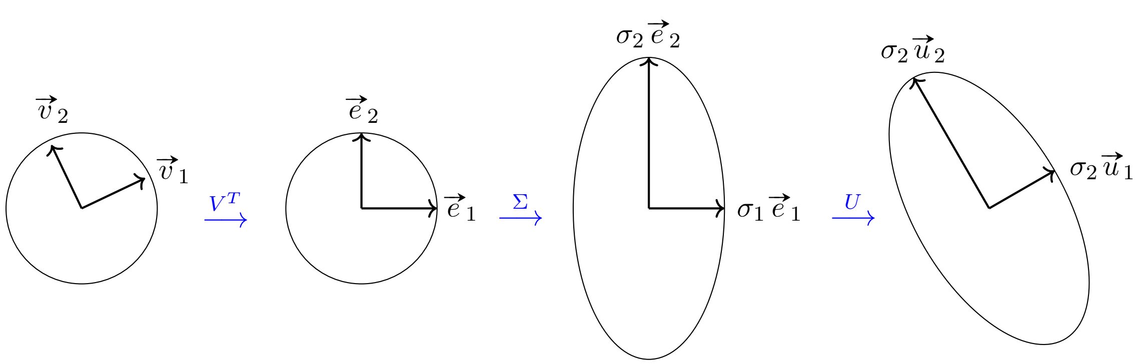
6.2.1 The Geometry of the Singular Value Decomposition

In this section we will work over $\mathbb{F} = \mathbb{R}$.

If we diagonalize a square matrix $A \in M_{n\times n}(\mathbb{R})$ to get $A = PDP^{-1}$, then we can interpret this factorization as follows. The matrix A sends the vector $\vec{x} \in \mathbb{R}^n$ to the vector $A\vec{x} = PDP^{-1}\vec{x}$. The matrix P^{-1} changes coordinates from the standard basis for $\mathbb{R}^n$ to our basis of eigenvectors, then the matrix D scales the coordinate of each eigenvector by the corresponding eigenvalue, and finally P converts the result back to the standard basis.

If we can *orthogonally* diagonalize A, then P and $P^{-1} = P^T$ are still change of basis matrices, but they change bases from the one orthonormal basis to another. In a sense, they are built up of rotations and reflections, but they don't perform any "stretching."

Now if $A \in M_{m\times n}(\mathbb{R})$ has an SVD given by $A = U\Sigma V^T$, then we can interpret the orthogonal matrices V^T and U as each performing rotations and/or reflections, while the diagonal matrix Σ performs a scaling (possibly by zero in some directions). So the singular value decomposition now paints the following picture of any linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$: it is built up of rotations and/or reflections, followed by scalings, then followed by additional rotations and/or reflections! For instance, the effect of $A \in M_{2\times 2}(\mathbb{R})$ on the unit circle can be pictured as follows:



6.3 Singular Value Decomposition of Linear Maps

We now come to the result that we wished to establish in the introduction to this chapter. We're going to prove that a linear map between finite-dimensional inner product spaces can always be "diagonalized" by picking appropriate (and possibly distinct) orthonormal bases for its domain and codomain.

Theorem 6.3.1 **(Singular Value Decomposition of Linear Maps)**

Let $L\colon V \to W$ be a linear map between finite-dimensional inner product spaces of dimensions n and m, respectively. If $r = \operatorname{rank}(A)$, then there exist orthonormal basis $\mathcal{B}$ and $\mathcal{C}$ for V and W and an $r \times r$ diagonal matrix D such that

$$ {}_{\mathcal{C}}[L]_{\mathcal{B}} = \begin{bmatrix} D & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(n-r)\times(m-r)} \end{bmatrix}. $$

Proof: Choose orthonormal bases $\mathcal{B}'$ and $\mathcal{C}'$ for V and W, let $A = {}_{\mathcal{C}'}[L]_{\mathcal{B}'}$, and let $A = U\Sigma V^*$ be an SVD of A. Since $\mathcal{B}'$ is an orthonormal basis, the discussion preceding Theorem 5.4.6 shows that the inner product of two vectors in V is the same as the inner product of their $\mathcal{B}'$-coordinate vectors; and the analogous statement is true for W and $\mathcal{C}'$. Thus if we let $\mathcal{B}$ be the basis for V consisting of the vectors whose $\mathcal{B}'$-coordinates are the columns of V, and if we let $\mathcal{C}$ be the basis for W consisting of the vectors whose $\mathcal{C}'$-coordinates are the columns of U, then $\mathcal{B}$ and $\mathcal{C}$ are orthonormal bases, and we have $U = {}_{\mathcal{C}'}\mathcal{I}_{\mathcal{C}}$ and $V = {}_{\mathcal{B}'}\mathcal{I}_{\mathcal{B}}$ so that

$$ {}_{\mathcal{C}}[L]_{\mathcal{B}} = {}_{\mathbb{C}}\mathcal{I}_{\mathcal{C}'}\ {}_{\mathcal{C}'}[L]_{\mathcal{B}'}\ {}_{\mathcal{B}'}\mathcal{I}_{\mathcal{B}} = U^{-1}AV = \Sigma, $$

as required. □

Let's see the proof of this theorem in action.

Example 6.3.2 Consider the differentiation map $D\colon \mathcal{P}_2(\mathbb{F}) \to \mathcal{P}_1(\mathbb{F})$ given by $D(p(x)) = p'(x)$, and suppose that both polynomial spaces are endowed with the inner product $\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)\,dx$.

The first order of business is to find orthonormal bases $\mathcal{B}'$ and $\mathcal{C}'$ for $\mathcal{P}_2(\mathbb{R})$ ad $\mathcal{P}_1(\mathbb{R})$, respectively. According to Example 4.3.6, we can take

$$ \mathcal{B}' = \left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}x, \sqrt{\frac{5}{2}}\left(\frac{3}{2}x^2 - \frac{1}{2}\right) \right\} \quad \text{and} \quad \mathcal{C}' = \left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}x \right\}. $$

Then let

$$ A = {}_{\mathcal{C}'}[D]_{\mathcal{B}'} = \begin{bmatrix} 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{15} \end{bmatrix}. $$

We must now find an SVD of A. We have

$$ A^TA = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}. $$

So the eigenvalues of A^*A are $\lambda_1 = 15$, $\lambda_2 = 3$ and $\lambda_3 = 0$ with corresponding eigenvectors $\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Thus the singular values of A are $\sigma_1 = \sqrt{15}$, $\sigma_2 = \sqrt{3}$ and $\sigma_3 = 0$, with corresponding singular vectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ as above.

Now let

$$\vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{u}_2 = \frac{1}{\sigma_2} A\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Then $\{\vec{u}_1, \vec{u}_2\}$ is an orthonormal basis for $\mathbb{R}^2$. So if we let

$$V = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

then the corresponding SVD of A is

$$A = U \begin{bmatrix} \sqrt{15} & 0 & 0 \\ 0 & \sqrt{3} & 0 \end{bmatrix} V^T.$$

Let's convert everything back to polynomials. Our desired basis $\mathcal{B}$ for $\mathcal{P}_2(\mathbb{R})$ comes from the columns of V (thought of as being $\mathcal{B}'$-coordinate vectors):

$$\mathcal{B} = \left\{ \sqrt{\frac{5}{2}} \left(\frac{3}{2}x^2 - \frac{1}{2} \right), \sqrt{\frac{3}{2}}x, \frac{1}{\sqrt{2}} \right\}.$$

Similarly, our desired basis $\mathcal{C}$ for $\mathcal{P}_1(\mathbb{R})$ comes from the columns of U (thought of as being $\mathcal{C}'$-coordinate vectors):

$$\mathcal{C} = \left\{ \sqrt{\frac{3}{2}}x, \frac{1}{\sqrt{2}} \right\}.$$

We'll leave it to you to check that

$${}_{\mathcal{C}}[D]_{\mathcal{B}} = \begin{bmatrix} \sqrt{15} & 0 & 0 \\ 0 & \sqrt{3} & 0 \end{bmatrix},$$

as claimed by Theorem 6.3.1 (bear in mind that $r = \text{rank}(D) = 2$).

6.4 Applications of the Singular Value Decomposition

In this final section we will highlight a select few applications of the singular value decomposition. We will barely scratch the surface of what is possible. Indeed, the applications of the singular value decomposition are too numerous and too broad, ranging from computational mathematics, statistics, text analysis, image and data compression, machine learning, quantum physics, to even psychology! Essentially, if you are able to put your data into a matrix somehow, then it's very likely that the SVD will say something interesting about this matrix (and hence your data).

Here's a simple, but relatable, example. You've probably used software to compute the rank or nullspace of a matrix. Have you wondered how the software does these computations? You might be surprised to learn that it very likely used SVD at some point. For instance, Mathematica (hence WolframAlpha), Maple and MATLAB all use SVD in their rank and nullspace routines. One reason for this is that SVD offers certain numerical advantages that makes it more stable than, say, the more efficient Gaussian elimination algorithm.

6.4.1 Low-rank Approximations

Many real-life applications of linear algebra involve large matrices built out of data. For instance, the results of a demographic survey of the population of Ontario can be recorded in a matrix each of whose columns represents a resident of Ontario, and whose rows represent values such as age, gender, income and city of residence. Movies on a streaming platform could be represented as rows in a matrix whose columns contain information such as language, running time and genre. A gray-scale image could be represented as a matrix whose entries are the color intensities of each pixel.

In practice, these large data matrices tend to contain certain dominant features due to the inherent correlation in the data (e.g. nearby pixels in an image tend to have the same shade of colour), and it's highly desirable that these dominant features be identified. The SVD provides us with a way to do this. It turns out that the singular vectors associated to the larger singular values contain most of the information about the matrix, in a certain sense. Therefore by "forgetting" about the parts of the matrix coming from the smaller singular values, we're able to somehow compress our matrix down into something that is simpler to analyze but that is still fairly representative. We will explain how this works in this section, and at the end we will illustrate by showing how these ideas can help with image compression.

As a first step, we will introduce a more compact version of SVD that removes unnecessary rows and columns from U, Σ and V^T. Let's look back at the SVDs from Examples 6.2.4–6.2.6 to illustrate the idea. We had obtained the decompositions

$$\begin{bmatrix} 1 & -4 \\ -2 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{v}_1^T \\ \vec{v}_2^T \end{bmatrix} \tag{6.1}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} \vec{v}_1^T \\ \vec{v}_2^T \\ \vec{v}_3^T \end{bmatrix} \tag{6.2}$$

$$\begin{bmatrix} 0 & 2 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \vec{v}_1^T \\ \vec{v}_2^T \end{bmatrix}. \tag{6.3}$$

In (6.1), the row of 0s in Σ indicates that the vector $\vec{u}_3$ is rather unnecessary since it will get multiplied by 0. So if we delete $\vec{u}_3$ and the row of 0s from Σ, we would obtain the simpler decomposition

$$\begin{bmatrix} 1 & -4 \\ -2 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \vec{v}_1^T \\ \vec{v}_2^T \end{bmatrix}$$

Likewise, the column of 0s in (6.2) indicates that we could've done without $\vec{v}_3^T$. By deleting both, we obtain the simpler decomposition

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \vec{v}_1^T \\ \vec{v}_2^T \end{bmatrix}.$$

On the other hand, there are no redundancies in (6.3).

In general, suppose that A is an $m \times n$ matrix of rank r with SVD $A = U\Sigma V^*$. The $m \times n$ matrix Σ will have r non-zero entries on its diagonal. By deleting all zero rows and columns from Σ, we are left with an $r \times r$ diagonal matrix $\Sigma_r = \operatorname{diag}(\sigma_1, \ldots, \sigma_r)$. Let U_r and V_r be the matrices formed from the first r columns of U and V. Thus, U_r is an $m \times r$ matrix and V_r^* is an $r \times n$ matrix.

Definition 6.4.1
Compact SVD

Let $A \in M_{m\times n}(\mathbb{F})$ be a rank r matrix with singular value decomposition $A = U\Sigma V^*$. Let Σ_r, U_r and V_r be as desribed in the preceding paragraph. The decomposition

$$A = U_r \Sigma_r V_r$$

is called a **compact** singular value decomposition of A.

Example 6.4.2

In Example 6.2.7 we obtained the singular value decomposition

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \\ i & 2i & -i \end{bmatrix} = U \begin{bmatrix} \sqrt{42} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^*,$$

where

$$U = \begin{bmatrix} -\frac{1}{\sqrt{7}} & \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{21}} \\ \frac{-2}{\sqrt{7}} & 0 & 0 & \frac{3}{\sqrt{21}} \\ \frac{1}{\sqrt{7}} & 0 & \frac{2}{\sqrt{6}} & \frac{2}{\sqrt{21}} \\ -\frac{i}{\sqrt{7}} & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{6}} & -\frac{2i}{\sqrt{21}} \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ -\frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}.$$

The matrix A has rank $r = 1$ (since it has only one non-zero singular value). Let's delete the last three zero rows from Σ (hence the last three columns from U), and then let's delete the last two columns from Σ (hence the last two columns from V). We are left with

$$\Sigma_r = \begin{bmatrix} \sqrt{42} \end{bmatrix}, \quad U_r = \frac{1}{\sqrt{7}} \begin{bmatrix} -1 \\ -2 \\ 1 \\ -i \end{bmatrix} \quad \text{and} \quad V_r = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}.$$

The corresponding compact SVD is

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \\ i & 2i & -i \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{7}} \\ -\frac{2}{\sqrt{7}} \\ \frac{1}{\sqrt{7}} \\ -\frac{i}{\sqrt{7}} \end{bmatrix} \begin{bmatrix} \sqrt{42} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}.$$

You should multiply out the right side to confirm that it does indeed give the left side!

Notice that we've replaced our original decomposition $A = U\Sigma V^*$ with the much simpler $A = \sigma_1 \vec{u}_1 \vec{v}_1^*$.

The compact SVD in the example above took on the particularly simple form $A = \sigma_1 \vec{u}_1 \vec{v}_1^*$ owing to the fact that $r = \text{rank}(A)$ was equal to 1. In the general situation, we'll be able to express A as a sum of r matrices of this form.

Proposition 6.4.3 Let $A \in M_{m\times n}(\mathbb{F})$ have rank r and compact SVD $A = U_r \Sigma_r V_r^*$, where $\Sigma_r = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_r \end{bmatrix}$. Let $\vec{u}_1, \ldots, \vec{u}_r$ and $\vec{v}_1, \ldots, \vec{v}_r$ be the columns of U_r and V_r, respectively. Then:

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^* + \cdots + \sigma_r \vec{u}_r \vec{v}_r^*.$$

Proof: We have

$$\begin{aligned} A &= U_r \Sigma_r V_r^* \\ &= \begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_r \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_r \end{bmatrix} \begin{bmatrix} \vec{v}_1^* \\ \vdots \\ \vec{v}_r^* \end{bmatrix} \\ &= \begin{bmatrix} \sigma_1 \vec{u}_1 & \cdots & \sigma_r \vec{u}_r \end{bmatrix} \begin{bmatrix} \vec{v}_1^* \\ \vdots \\ \vec{v}_r^* \end{bmatrix} \\ &= \sigma_1 \vec{u}_1 \vec{v}_1^* + \cdots + \sigma_r \vec{u}_r \vec{v}_r^*, \end{aligned}$$

using the definition of matrix multiplication. □

Example 6.4.4 From Example 6.2.4, we have the SVD

$$\begin{bmatrix} 1 & -4 \\ -2 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

hence compact SVD

$$\begin{bmatrix} 1 & -4 \\ -2 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} 1 & -4 \\ -2 & 2 \\ 2 & 4 \end{bmatrix} = 6 \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} + 3 \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

EXERCISE

Let $\vec{u}, \vec{v} \in \mathbb{F}^n$. Show that the $n \times n$ matrix $\vec{u}\vec{v}^*$ has rank 1. This is sometimes called the **outer product** of $\vec{u}$ and $\vec{v}$. (Recall that the standard *inner* product of $\vec{u}$ and $\vec{v}$ is given by $\vec{v}^*\vec{u}$.)

In view of the above exercise and Proposition 6.4.3, we're now able to use the SVD to express a given rank r matrix A as the sum of r matrices of rank 1:

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^* + \cdots + \sigma_r \vec{u}_r \vec{v}_r^*.$$

Since the singular values of A are ordered in descending order $\sigma_1 \geq \cdots \geq \sigma_r$, the tail-end of this representation will in some sense be "less significant" if the smaller singular values are very small. In many real-life scenarios, the small singular values will be numerically much smaller than the first few large singular values, and so if we discard them then not much is lost. This motivates the following definition.

Definition 6.4.5
Rank-k Truncation

Let $A \in M_{m\times n}(\mathbb{F})$ be a rank r matrix with singular values $\sigma_1 \geq \cdots \geq \sigma_r > 0$ and compact singular value decomposition $A = U_r \Sigma_r V_r^*$, where $U = \begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_r \end{bmatrix}$ and $V = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_r \end{bmatrix}$. Let $k \leq r$ be a positive integer. The **rank-k truncation** of A is

$$A_k = \sigma_1 \vec{u}_1 \vec{v}_1^* + \cdots + \sigma_k \vec{u}_k \vec{v}_k^*.$$

That is, the rank-k truncation only keeps the contributions of the largest k singular values and corresponding singular vectors and discards the rest. Here are some key properties.

Proposition 6.4.6

Let $A \in M_{m\times n}(\mathbb{F})$ be a matrix of rank r with rank-k truncation

$$A_k = \sigma_1 \vec{u}_1 \vec{v}_1^* + \cdots + \sigma_k \vec{u}_k \vec{v}_k^*.$$

Then:

(a) $A = A_r$.

(b) $\operatorname{rank}(A_k) = k$.

(c) $\|A - A_k\| \leq \sum_{i=k+1}^{r} \sigma_i$, where $\|X\| = \sqrt{\operatorname{tr}(X^*X)}$ is the norm induced from the inner product $\langle A, B\rangle = \operatorname{tr}(B^*A)$ on $M_{m\times n}(\mathbb{F})$. (See Example 4.1.6 and the two exercises that follow it.)

Proof: (a) This is just Proposition 6.4.3.

(b) Let's view A_k as giving the linear map $L_k \colon \mathbb{F}^n \to \mathbb{F}^m$ defined by $L_k(\vec{x}) = A_k\vec{x}$. We claim that the range of L_k is $\operatorname{Span}\{\vec{u}_1, \ldots, \vec{u}_k\}$. Since the vectors $\{\vec{u}_1, \ldots, \vec{u}_k\}$ are linearly independent, this will show that the rank of L_k, hence of A_k, is equal to k, as required.

To see why the claim about the range of L_k is true, consider first any $\vec{x} \in \mathbb{F}^n$. Then

$$L_k(\vec{x}) = A_k\vec{x} = \sum_{i=1}^{k} \sigma_i \vec{u}_i \vec{v}_i^* \vec{x} = \sum_{i=1}^{k} \sigma_i \vec{u}_i \langle \vec{x}, \vec{v}_i\rangle$$

is a linear combination of $\{\vec{u}_1, \ldots, \vec{u}_k\}$, proving that $\operatorname{Range}(L_k) \subseteq \operatorname{Span}\{\vec{u}_1, \ldots, \vec{u}_k\}$. Conversely, given any $j \in \{1, \ldots, k\}$, consider

$$L_k(\vec{v}_j) = \sum_{i=1}^{k} \sigma_i \vec{u}_i \langle \vec{v}_j, \vec{v}_i\rangle .$$

Since $\{\vec{v}_1, \ldots, \vec{v}_j\}$ is an orthonormal set, the above reduces to $L_k(\vec{v}_j) = \sigma_j \vec{u}_j$. And since $\sigma_j \neq 0$ by the definition of the compact SVD, it follows that $\vec{u}_j = L_k(\frac{1}{\sigma_j}\vec{v}_j)$ is in the range of L_k. This completes the proof that $\text{Range}(L_k) = \text{Span}\{\vec{u}_1, \ldots, \vec{u}_r\}$, as claimed.

(c) We have

$$\begin{aligned}
\|A - A_k\| &= \left\| \sum_{i=k+1}^{r} \sigma_i \vec{u}_i \vec{v}_i^* \right\| \\
&\leq \sum_{i=k+1}^{r} |\sigma_i| \, \|\vec{u}_i \vec{v}_i^*\| \\
&= \sum_{i=k+1}^{r} \sigma_i \operatorname{tr}\left((\vec{u}_i \vec{v}_i^*)^* \vec{u}_i \vec{v}_i^*\right) \\
&= \sum_{i=k+1}^{r} \sigma_i \operatorname{tr}\left(\vec{v}_i \vec{u}_i^* \vec{u}_i \vec{v}_i^*\right) \\
&= \sum_{i=k+1}^{r} \sigma_i \operatorname{tr}\left(\vec{v}_i \vec{v}_i^*\right),
\end{aligned}$$

where the last equality follows since $\vec{u}_i^* \vec{u}_i = \langle \vec{u}_i, \vec{u}_i \rangle = 1$. Now, using the fact that $\text{tr}(XY) = \text{tr}(YX)$ for matrices X and Y of compatible sizes, we see that $\operatorname{tr}(\vec{v}_i \vec{v}_i^*) = \operatorname{tr}(\vec{v}_i^* \vec{v}_i) = \langle \vec{v}_i, \vec{v}_i \rangle = 1$, completing the proof.

□

EXERCISE

We can do better than the inequality in part (c) of Proposition 6.4.6. Show that, in fact,

$$\|A - A_k\| = \sqrt{\sum_{i=k+1}^{r} \sigma_i^2}.$$

[**Hint:** Show that $\vec{u}_i \vec{v}_i^* \perp \vec{u}_j \vec{v}_j^*$.]

Part (b) of Proposition 6.4.6 justifies that the name "rank-k truncation" for A_k, while part (c) shows that if the singular values σ_i for $i > k$ are small, then $\|A - A_k\|$ will be small too, and so A_k will in this sense be a good approximation to A. Of course, this is only a heuristic justification, since even if the σ_i's are small, their sum might not be small. However, there is a sense in which A_k is *the best* rank k approximation to A. This is the content of the next theorem, which we state without proof. (The proof is not difficult, but requires a few extra ideas that will veer us off course.)

Theorem 6.4.7 **(Eckart–Young Theorem)**

Let $A \in M_{m\times n}(\mathbb{F})$, and let A_k be the rank-k truncation of A. Let $B \in M_{m\times n}(\mathbb{F})$ be an arbitrary rank k matrix. Then

$$\|A - B\| \geq \|A - A_k\|.$$

Thus, A_k is closer to A than any other rank k matrix B.

Example 6.4.8 Consider the matrix $A = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & 3 \\ -1 & 0 & 1 \\ -4 & 0 & 2 \end{bmatrix}$. Using software, we find $A = U\Sigma V^*$ with

$$U \approx \begin{bmatrix} -0.408 & -0.207 & -0.833 & -0.312 \\ 0.272 & 0.894 & -0.356 & 0 \\ 0.274 & 0.003 & 0.217 & -0.937 \\ 0.827 & -0.397 & -0.365 & 0.156 \end{bmatrix}, \quad V \approx \begin{bmatrix} -0.721 & 0.677 & 0.148 \\ 0.053 & 0.266 & -0.962 \\ 0.691 & 0.686 & 0.228 \end{bmatrix}$$

and

$$\Sigma \approx \begin{bmatrix} 5.156 & 0 & 0 \\ 0 & 3.358 & 0 \\ 0 & 0 & 0.370 \\ 0 & 0 & 0 \end{bmatrix}.$$

We can delete the bottom row from Σ and the fourth column from U to obtain a compact SVD. Here are the rank truncations of A:

$$A_1 \approx 5.156 \begin{bmatrix} -0.408 \\ 0.272 \\ 0.274 \\ 0.827 \end{bmatrix} \begin{bmatrix} -0.721 & 0.053 & 0.691 \end{bmatrix} = \begin{bmatrix} 1.517 & -0.111 & -1.452 \\ -1.011 & 0.074 & 0.968 \\ -1.019 & 0.075 & 0.975 \\ -3.074 & 0.226 & 2.943 \end{bmatrix}$$

$$A_2 \approx A_1 + 3.358 \begin{bmatrix} -0.207 \\ 0.894 \\ 0.003 \\ -0.397 \end{bmatrix} \begin{bmatrix} 0.677 & 0.266 & 0.686 \end{bmatrix} = \begin{bmatrix} 1.046 & -0.295 & -1.930 \\ 1.021 & 0.872 & 3.028 \\ -1.012 & 0.077 & 0.983 \\ -3.977 & -0.131 & 2.032 \end{bmatrix}$$

$$A_3 = A.$$

We have $\|A - A_1\| \approx 3.379 \leq \sigma_2 + \sigma_3$ and $\|A - A_2\| \approx 0.36983 \leq \sigma_3$, as predicted by Proposition 6.4.6(c).

Let's now illustrate how all this can be applied to perform image compression. The basic idea is to encode an image into a matrix, say by converting to gray-scale and then recording the (i, j)th pixel intensity as the (i, j)th entry of a matrix A. This will usually create a rather large matrix, with large rank. However, in practice, most of the singular values of A will be extremely small. Thus we will be able to use the rank-k approximation for $k \ll \text{rank}(A)$ to approximate the original image. This results in us having to use less memory to store the image, since we only need to use the truncated SVD data to reconstruct it.

Here is a simple example of three rank-k truncations of an image.

(a) $k = 5$ (b) $k = 25$ (c) $k = 75$ (d) Original ($r = 361$)

Figure 6.4.1: Low-rank image compression

We should mention in closing that this is not a very sophisticated image compression technique (in particular, it is *lossy*), and there are many better compression algorithms available. Nonetheless, it is a very compelling application of SVD! You can visit Tim Baumann's SVD-Demo webpage to try it yourself.

6.4.2 The Pseudoinverse and Least Squares Revisited

Let's consider once more the basic problem of solving a system of equations given in matrix form as $A\vec{x} = \vec{b}$, where $A \in M_{m\times n}(\mathbb{F})$ and $\vec{b} \in \mathbb{F}^m$. As you well know, this system has a solution if and only if $\vec{b} \in \operatorname{Col}(A)$. If A is square and invertible, then this condition is always satisfied, and in fact we know that the system will have a *unique* solution, which is given by $\vec{x} = A^{-1}\vec{b}$.

If A is an arbitrary $m \times n$ matrix, we can use the SVD $A = U\Sigma V^*$ to re-write the equation $A\vec{x} = \vec{b}$ as

$$U\Sigma V^* \vec{x} = \vec{b}.$$

We now wish to "invert" U, Σ and V. With U and V, this is no problem, since they are square and unitary. However, Σ is an $m \times n$ matrix, so it doesn't make sense to invert it. But let's pretend we can: the non-zero diagonal entries of Σ are the non-zero singular values $\sigma_1, \ldots, \sigma_r$, so let's define $\Sigma^\dagger$ to be the $n \times m$ matrix (not the $m \times n$ matrix!) whose diagonal entries are $\frac{1}{\sigma_1}, \ldots, \frac{1}{\sigma_r}$ and all of whose other entries are 0.

Example 6.4.9

If $\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ then $\Sigma^\dagger = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$. Notice that in this case $\Sigma^\dagger$ is equal to Σ^{-1}. On the other hand,

$$\begin{bmatrix} 6 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}^\dagger = \begin{bmatrix} \frac{1}{6} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}^\dagger = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{bmatrix}.$$

Now, given the equation

$$U\Sigma V^* \vec{x} = \vec{b}$$

we can "invert" each matrix on the left-side, in order, to get the "solution"

$$\vec{x}_0 = V\Sigma^\dagger U^* \vec{b}.$$

So, in effect, the matrix $V\Sigma^\dagger U^*$ is acting like some kind of inverse for $A = U\Sigma V^*$. We'll explain the meaning of this $\vec{x}_0$ below, but first let's introduce some terminology.

Definition 6.4.10
Pseudoinverse

Let $A \in M_{m\times n}(\mathbb{F})$ be a matrix of rank r with SVD $A = U\Sigma V^*$. The **pseudoinverse** of A is the $n \times m$ matrix

$$A^\dagger = V\Sigma^\dagger U^*,$$

where $\Sigma^\dagger$ is the $n \times m$ matrix whose (i, j)th entry is $\frac{1}{\sigma_i}$ for $i = j \leq r$ and 0 otherwise.

Notice that $\Sigma^\dagger$ is in fact the pseudoinverse of Σ, so the notation is consistent. The pseudoinverse of A is uniquely determined by A, and does not depend on the choice of singular value decomposition (i.e. on a choice of U and V).

Example 6.4.11

If A is an invertible $n \times n$ matrix, then all n of its singular values are nonzero thanks to Proposition 6.1.9. Thus in this case $\Sigma^\dagger = \Sigma^{-1}$ and it follows that the pseudoinverse of A is actually the inverse of A:

$$A^\dagger = A^{-1}.$$

EXERCISE

Supply the remaining details to prove that if A is invertible then $A^\dagger = A^{-1}$.

Example 6.4.12

For $A = \begin{bmatrix} 1 & -4 \\ -2 & 2 \\ 2 & 4 \end{bmatrix}$ as in Example 6.2.4, we found that $A = U\Sigma V^*$ with

$$U = \frac{1}{3}\begin{bmatrix} -2 & 1 & 2 \\ 1 & -2 & 2 \\ 2 & 2 & 1 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 6 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Thus,

$$\begin{aligned} A^\dagger &= V\Sigma^\dagger U^* \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{6} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix} \frac{1}{3} \begin{bmatrix} -2 & 1 & 2 \\ 1 & -2 & 2 \\ 2 & 2 & 1 \end{bmatrix} \\ &= \frac{1}{18}\begin{bmatrix} 2 & -4 & 4 \\ -2 & 1 & 2 \end{bmatrix}. \end{aligned}$$

Let's return now to our system of linear equations

$$A\vec{x} = \vec{b}, \tag{6.4}$$

where $A \in M_{m\times n}(\mathbb{F})$ and $b \in \mathbb{F}^m$. Using the pseudoinverse of A, we create the vector $\vec{x}_0 = A^\dagger \vec{b} \in \mathbb{F}^n$. If A is square an invertible, then $\vec{x}_0 = A^{-1}\vec{b}$ is the unique solution to the system (6.4). In the general case, $\vec{x}_0$ is still interesting, as the next proposition explains.

Proposition 6.4.13 **(Minimal Norm Solutions)**

Consider the system of linear equations $A\vec{x} = \vec{b}$, where $A \in M_{m\times n}(\mathbb{F})$ and $\vec{b} \in \mathbb{F}^m$. Let $\vec{x}_0 = A^\dagger \vec{b}$. Then:

(a) If $A\vec{x} = \vec{b}$ is consistent, then $\vec{x}_0$ is a solution to the system. Moreover, it is the solution of minimal norm, i.e., if $\vec{x}$ is any solution to the system, then $\|\vec{x}\| \geq \|\vec{x}_0\|$ with equality if and only if $\vec{x} = \vec{x}_0$.

(b) If $A\vec{x} = \vec{b}$ is inconsistent, then $\vec{x}_0$ is a least squares solution (Definition 4.7.1). Moreover, it is the least squares solution of minimal norm, i.e., if $\vec{s}$ is any least squares solution to the system, then $\|\vec{s}\| \geq \|\vec{x}_0\|$ with equality if and only if $\vec{s} = \vec{x}_0$.

REMARK

Although in our initial discussion of least squares (Section 4.7) we only worked over $\mathbb{F} = \mathbb{R}$, everything works just as well over $\mathbb{F} = \mathbb{C}$. The one difference is: instead of using transposes, we should be using conjugate-transposes, as you probably would have guessed.

Proof of Proposition 6.4.13: Let's show that $\vec{x}_0$ is always a least squares solution. In the case where the system is consistent, this will automatically imply that $\vec{x}_0$ is an actual solution (why?). By Proposition 4.7.3, we must show that $A^*A\vec{x}_0 = A^*\vec{b}$. We have

$$\begin{aligned} A^*A\vec{x}_0 &= A^*AA^\dagger\vec{b} \\ &= (V\Sigma^*U^*)(U\Sigma V^*)(V\Sigma^\dagger U^*)\vec{b} \\ &= V\Sigma^*\Sigma\Sigma^\dagger U^*\vec{b}. \end{aligned} \tag{$*$}$$

Now, the matrix $\Sigma\Sigma^\dagger$ will take the form $\begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, with enough 0s to fill out an $m\times m$ matrix. On the other hand, Σ^* will take the form $\begin{bmatrix} D^* & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, where $D = \operatorname{diag}(\sigma_1, \ldots, \sigma_r)$ is $r\times r$ and with enough 0s to fill out an $n \times m$ matrix. It follows that

$$\Sigma^*\Sigma\Sigma^\dagger = \begin{bmatrix} D^* & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} D^* & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \Sigma^*.$$

So from $(*)$ we obtain

$$A^*A\vec{x}_0 = V\Sigma^*U^*\vec{b} = A^*\vec{b},$$

proving that $\vec{x}_0$ is a least squares solution to $A\vec{x} = \vec{b}$, as desired. All that remains now is proving that $\vec{x}_0$ has minimal norm.

If $A\vec{x} = \vec{b}$ is consistent, then an arbitrary solution will take the form $\vec{x}_0 + \vec{z}$ where $\vec{z} \in \mathrm{Null}(A)$, since $\vec{x}_0$ is a solution. Now we make two observations. First, $\vec{z} \in \mathrm{Col}(A^*)^\perp$, by Proposition 6.2.3. Next, $\vec{x}_0 = A^\dagger \vec{b} = V(\Sigma^\dagger U^* \vec{b})$ is a linear combination of the first r columns of V, which are eigenvectors of A^*A corresponding to nonzero eigenvalues; so each of these eigenvectors is in $\mathrm{Col}(A^*)$ (since if $A^*A\vec{v} = \lambda\vec{v}$ with $\lambda \neq 0$ then $\vec{v} = A^*(\frac{1}{\lambda}A\vec{v})$), and therefore $\vec{x}_0 \in \mathrm{Col}(A^*)$. Combining both observations, we deduce that $\vec{x}_0 \perp \vec{z}$. Hence, by the Pythagorean theorem,

$$\|\vec{x}_0 + \vec{z}\|^2 = \|\vec{x}_0\|^2 + \|\vec{z}\|^2 \geq \|\vec{x}_0\|^2,$$

completing the proof of the minimality of $\|\vec{x}_0\|$ amongst norms of solutions. Notice also that the above inequality is an equality if and only if $\vec{z} = \vec{0}$.

Finally, if $A\vec{x} = \vec{b}$ is inconsistent, then an arbitrary least squares solution will take the form $\vec{x}_0 + \vec{z}$ where $\vec{z} \in \mathrm{Null}(A)$. The same proof of minimality above, which never used the fact that $\vec{x}_0$ was an actual solution to $A\vec{x} = \vec{b}$, applies once more to give us the desired minimality result in this case. This completes the proof of the proposition. □

Example 6.4.14

Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ and consider the system $A\vec{x} = \vec{b}$ where $\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. In Example 4.7.2 we noted that this system is inconsistent, and we determined the set of least squares solutions to be

$$S = \left\{ \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}.$$

The norm of an arbitrary vector in S is given by

$$\left\| \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\| = \sqrt{\left(\frac{1}{2} - 2t\right)^2 + t^2}.$$

Using calculus, it's easy to show that this norm is minimized precisely when $t = \frac{1}{5}$. The corresponding least squares solution is $\frac{1}{10}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Let's confirm that this is inline with Proposition 6.4.13. An SVD for A is $A = U\Sigma V^*$, where

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}.$$

Hence the pseudoinverse of A is

$$A^\dagger = V\Sigma^\dagger U^* = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{10}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}.$$

Thus, the minimal norm least squares solution should be

$$A^\dagger = A^\dagger \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

just as we've computed above!

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